Last week: $\mathring{B}^u = B^- u \cdot B^+$, $\mathring{B}_w = B^+ w \cdot B^+$

$$\mathring{B}_{u,w} = \mathring{B}^u \cap \mathring{B}_w$$

<u>Thm</u>: TFAE

(1) $\mathring{B}_{u,w} \neq \phi$

(2) $B_{u,w} \neq \phi$

(3) $u \leq w$

We have proved (1) $\Rightarrow$ (2) $\Rightarrow$ (3)

Now: (3) $\Rightarrow$ (1)

<u>Proof</u>: We first show (1) is equivalent to:

$$\mathring{B}^u \cap (w U^- B^+) \neq \phi$$

In fact,

$$\begin{aligned} wU^-B^+ = {}^wU^- wB &= ({}^wU^- \cap U^-) \times \left(({}^wU^- \cap U^+) \cdot B^+\right) \\ &= ({}^wU^- \cap U^-) \times \mathring{B}_w \end{aligned}$$

Since $\mathring{B}^u$ is invariant under left action by ${}^wU^- \cap U^-$,

we have $\mathring{B}^u \cap (wU^-B^+) = ({}^wU^- \cap U^-) \times \mathring{B}_{u,w}$

We now show $\mathring{B}^u \cap (wU^-B^+) \neq \phi$

since wU^-B^+ is open, this is equivalent to

$$B^u \cap (wU^-B^+) \neq \phi$$

which follows if one observe $wB^+ \in B^u \cap (wU^-B^+)$

(use $u \leq w$ here)

Recall

1. Reduction map

$$B^+ \xrightarrow{v} \pi_v^w(B) \xrightarrow{v'} B$$

2. Deodhov decomposition

$\underline{w} = s_{i_1} \cdots s_{i_n}$ reduced expression

$\underline{v} = t_1 \cdots t_n$ subexpression ($t_j = 1$ or s_{i_j})

Set $v_{(j)} = t_1 \cdots t_j$

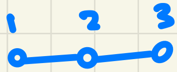
$\underline{v}$ is <u>distinguished</u> if $v_{(j)} \leq v_{(j-1)} s_{i_j}$

<u>positive</u> if $v_{(j-1)} < v_{(j-1)} s_{i_j}$

Here positive $\Longleftrightarrow$ distinguished + non-decreasing

Also, positive subexpressions are just the unique rightmost subexpressions

E.g. $S_4 = \langle s_1, s_2, s_3 \rangle$



$\underline{w} = s_3 s_2 s_1 s_3 s_2 s_3$ $\qquad v = s_2$

distinguished: $(1\ 1\ 1\ 1\ s_2\ 1)$ positive

$(s_3\ 1\ 1\ s_3\ s_2\ 1)$

$(s_3\ s_2\ 1\ s_3\ s_2\ s_3)$

$(1\ s_2\ 1\ s_3\ 1\ s_3)$

example of non-distinguished: $(1\ s_2\ 1\ 1\ 1\ 1)$

Deodhar Component (Fix a reduced $\underline{w}$ first)

Let $\underline{v}$ be a subexpression

Set $\mathring{B}_{\underline{v},\underline{w}} = \{ B \in \mathring{B}_{v,w} : \pi^{w}_{w_{(k)}}(B) \in \mathring{B}_{v_{(k)}, w_{(k)}} \ \forall k \}$

Notations:

$J_{\underline{v}}^{+} = \{ k \,;\, \underline{v}_{(k)} > \underline{v}_{(k-1)} \}$

$J_{\underline{v}}^{0} = \{ k \,;\, \underline{v}_{(k)} = \underline{v}_{(k-1)} \}$

$J_{\underline{v}}^{-} = \{ k \,;\, \underline{v}_{(k)} < \underline{v}_{(k-1)} \}$

- $\underline{v}$ positive $\Rightarrow J_{\underline{v}}^{-} = \phi$
- $\{1, \cdots, n\} = J_{\underline{v}}^{+} \amalg J_{\underline{v}}^{0} \amalg J_{\underline{v}}^{-}$
- $\#(J_{\underline{v}}^{+}) - \#(J_{\underline{v}}^{-}) = \ell(v)$

<u>Thm</u> (Deodhar)

(1) $\mathring{B}_{\underline{v},\underline{w}} \neq \phi \iff \underline{v}$ distinguished

(2) Over a field, if $\underline{v}$ distinguished,

then $\mathring{B}_{\underline{v},\underline{w}} \simeq (K^{\times})^{\#(J_{\underline{v}}^{0})} \times K^{\#(J_{\underline{v}}^{-})}$

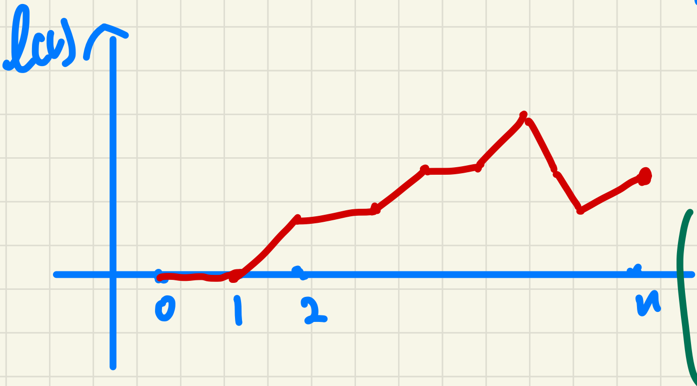
As a consequence,

$$\# \mathring{B}_{v,w}(\mathbb{F}_q) = \sum_{\underline{v} \text{ distinguished}} (q-1)^{\#(J_{\underline{v}}^{0})} \, q^{\#(J_{\underline{v}}^{-})}$$

$\uparrow$

Kazhan-Lustig's R polynomial

We first discuss some examples



$\dim \bar{B}_{\underline{v},\underline{w}} = \dim \mathring{B}_{v,w}$
for $\underline{v}$ positive

$\left(\because \text{ for } \underline{v} \text{ +ve, } J^- = \phi \right.$
$\# J^+ = \ell(v)$,
$\left. \# J^\circ = \ell(w) - \ell(v) = \dim \mathring{B}_{v,w} \right)$

E.g. $G = GL_2$, $w = s$ $\underline{w} = s$, $v = 1$, $\underline{v} = 1$

$$J^\circ_{\underline{v}} = \{1\}$$

$$\mathring{B}_{v,w} = \mathring{B}_{\underline{v},\underline{w}} = \{ y(a) \cdot B^+ : a \neq 0 \}$$

↑ as we have only one subexpression

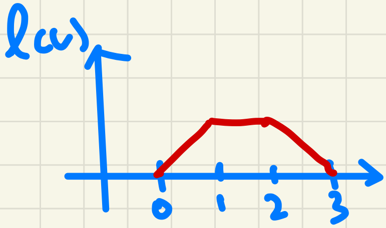
E.g. $G = GL_3$, $w = s_1 s_2 s_1$ $\underline{w} = s_1 s_2 s_1$, $v = 1$

$\underline{v}_1 = 111$ $J^\circ_{\underline{v}_1} = \{1, 2, 3\}$

$\mathring{B}_{\underline{v}_1,\underline{w}} = \{ y_1(a_1) y_2(a_2) y_1(a_3) : a_1 a_2 a_3 \neq 0 \}$ dim 3

$\underline{v}_2 = s_1 1 s_1$ $J^+_{\underline{v}_2} = \{1\}$, $J^\circ_{\underline{v}_2} = \{2\}$

$J^-_{\underline{v}_3} = \{3\}$



$$\mathring{B}_{\underline{v_3}, \underline{w}} = \{ \dot{s}_1 y_2(a) x_1(m) \dot{s}_1^{-1} : a \neq 0 \} \qquad \underline{\dim = 2}$$

$$= \{ y_{\alpha_1+\alpha_2}(a)\, y_1(m) \,;\, a \neq 0 \}$$

$$\dot{s}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$y_{\alpha_1+\alpha_2}(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix}$$

$$\dot{s}_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Marsh-Rietsch's formulation (§5)

Marsh-Rietsch parametrization:

Let $\underline{w} = s_{i_1} \cdots s_{i_n}$

Set

$$G_{\underline{v},\underline{w}} = \left\{ g = g_1 g_2 \cdots g_n \;\middle|\; \begin{array}{ll} g_k = x_{i_k}(m_k)\,\dot{s}_{i_k}^{-1}, & \text{if } k \in J_{\underline{v}}^{-},\ m_k \in K \\ g_k = y_{i_k}(t_k), & \text{if } k \in J_{\underline{v}}^{0},\ t_k \in K^{\times} \\ g_k = \dot{s}_{i_k}, & \text{if } k \in J_{\underline{v}}^{+} \end{array} \right.$$

$$\simeq (K^{\times})^{\#(J_{\underline{v}}^{0})} \times K^{\#(J_{\underline{v}}^{-})}$$

$$MR: G_{\underline{v},\underline{w}} \longrightarrow \mathring{B}_{v,w}, \quad g \mapsto g \cdot B^{+}$$

Thm: 1) $\mathcal{B}_{\underline{v},\underline{w}} = \phi$ if $\underline{v}$ not distinguished

2) MR gives $G_{\underline{v},\underline{w}} \xrightarrow{\sim} \mathring{B}_{\underline{v},\underline{w}}$ if $\underline{v}$ distinguished.

Proof: We prove the theorem by induction on $\ell(w)$.
The case $w = id$ is easy, so we may assume $\ell(w) > 0$.

Let $\underline{w} = s_{i_1} \cdots s_{i_n}$, $\underline{v} = t_1 \cdots t_n$

Denote $\underline{w}' = s_{i_1} \cdots s_{i_{n-1}}$ $\underline{v}' = t_1 \cdots t_{n-1}$

Suppose $\mathring{\mathcal{B}}_{\underline{v},\underline{w}} \neq \phi$, we will show

1) $\underline{v}$ is distinguished 2) $G_{\underline{v},\underline{w}} \xrightarrow{\sim} \mathring{\mathcal{B}}_{\underline{v},\underline{w}}$

Since $\mathring{\mathcal{B}}_{\underline{v}',\underline{w}'} \supset \pi^{w}_{w'}(\mathring{\mathcal{B}}_{\underline{v},\underline{w}}) \neq \phi$, we see by induction $\underline{v}'$ is distinguished.

Let $B \in \mathring{\mathcal{B}}_{\underline{v},\underline{w}}$, and $B' = \pi^{w}_{w'}(B) = g'B^+$, $g' \in G_{\underline{v}',\underline{w}'}$.

So $B = gB^+$ for some $g = g' x_{i_n}(m) \dot{s}_{i_n}^{-1} B^+$, $m \in k$.

There are several cases:

Case 1: $v' < v' s_{i_n}$

then $\underline{v}$ must be distinguished.

Case 1a): $m=0$

then $v = v' s_{i_n}$, $n \in J^+_{\underline{v}}$, $g = g' \dot{s}_{i_n}^{-1}$

Case 1b): $m \neq 0$

then $v = v'$, $n \in J^0_{\underline{v}}$, $g = g' x_{i_n}(m) \dot{s}_{i_n}^{-1}$

This case then follow from the SL_2 Calculations:

$$x(t)\dot{s} = \check{\alpha}(s)\, y(t)\, x(-t^{-1}) \qquad (\check{\alpha}(t) y(t) \check{\alpha}(t)^{-1} = y(t^{-1}))$$

$$\Rightarrow \quad g' x_{i_n}(t)\, \dot{s}_{i_n}^{-1} = g' y_{i_n}(t^{-1})\, B^+$$

Case 2 : $v' > v' s_{i_n}$

then $\mathring{B}^{v'} \cdot \mathring{B}^{s_{i_n}} = \mathring{B}^{v}$

hence $v = v' s_{i_n}$, $\underline{v}$ distinguished

$g = g' x_{i_n}(m)\, \dot{s}_{i_n}^{-1}$

It is also clear from the above Calculation that for each case, $G_{\underline{v},\underline{w}} \cdot B^+ \subset \mathring{B}_{\underline{v},\underline{w}}$

hence we are done.

Now back to TP. Our goals:

- $\mathcal{B}_{v,w>0} := \mathring{B}_{v,w} \cap \mathcal{B}_{\geq 0}$ is a connected component of $\mathring{B}_{v,w}(\mathbb{R})$

- $\overline{\mathcal{B}_{v,w>0}} = \coprod_{v \leq v' \leq w' \leq w} \mathcal{B}_{v',w'>0}$

$$\begin{tikzcd} B_{v,w>0} \arrow[d, hook] & G_{\underline{v_+},\underline{w}}(\mathbb{R}_{>0}) \arrow[l] \arrow[d, hook] \\ B_{\underline{v_+},\underline{w}}(\mathbb{R}) & G_{\underline{v_+},\underline{w}}(\mathbb{R}^\times) \arrow[l, "\simeq"'] \simeq (\mathbb{R}^\times)^{\ell(w)-\ell(v)} \end{tikzcd}$$

$\underline{v_+}$: the positive subexp. for

A decomposition of ${}^wU^-$. (after Kazhan-Lustig, Knutson-Woo-Yong)

$$g \in {}^wU^- \xleftarrow[\simeq]{\text{multi}} ({}^wU^- \cap U^-) \times ({}^wU^- \cap U^+) \ni (g_1, g_2)$$

$$\xleftarrow[\simeq]{\text{multi}} ({}^wU^- \cap U^+) \times ({}^wU^- \cap U^-) \ni (h_1, h_2)$$

Proposition: The map

$$C_w = (C_w^+, C_w^-) : {}^wU^- \longrightarrow ({}^wU^- \cap U^+) \times ({}^wU^- \cap U^-)$$

$$g \longmapsto (g_2, h_2)$$

is an isomorphism.

Proof: We will find an inverse map. i.e. for a given (g_2, h_2), need to find g_1, h_1

s.t. $\qquad h_1^{-1} g_1 = h_2 g_2^{-1}$

$\qquad h_1^{-1} \in {}^wU^- \cap U^+$

$\qquad g_1 \in {}^wU^- \cap U^-$

which determines g_1, h_1 uniquely.

Let $v \leq w$, choose $r \in W$

$$\begin{array}{ccccc} g \in {}^{r}U^{-} & \xrightarrow{c_r} & ({}^{r}U^{-} \cap U^{+}) \times ({}^{r}U^{-} \cap U^{-}) & \ni & (g_2, h_2) \\ \downarrow & \wr\wr & \downarrow & & \downarrow \\ g\dot{r}\cdot B^{+} \in & r U^{-}\cdot B^{+} \xrightarrow{\simeq} & U^{+}\dot{r}\cdot B^{+} \times U^{-}\dot{r}\cdot B^{+} & \ni & (g_2\dot{r}B^{+}, h_2\dot{r}\cdot B^{+}) \end{array}$$

$rU^{-}\cdot B^{+}$: open in $\mathcal{B}$, affine space $\simeq U^{-}$

$U^{+}\dot{r}\cdot B^{+} = \mathring{\mathcal{B}}_r$, $\dim \ell(r)$

$U^{-}\dot{r}\cdot B^{+} = \mathring{\mathcal{B}}^{r}$, $\operatorname{codim} \ell(r)$

<u>Q</u>: For $g \in {}^{r}U^{-}$ with $g\dot{r} \in B^{+}wB^{+} \cap B^{-}vB^{+}$

$g_2\dot{r} = ?$ $\qquad h_2\dot{r} = ?$

Here $g_2 \in U^{-}g$, $h_2 \in U^{+}g$.

So $g_2\dot{r} \in U^{-}g\dot{r} \subseteq B^{-}vB^{+}$

$h_2\dot{r} \in U^{+}g\dot{r} \subseteq B^{+}wB^{+}$

So $g_2\dot{r}\cdot B^{+} \in \mathring{\mathcal{B}}_{v,r}$, $h_2\dot{r}B^{+} \in \mathring{\mathcal{B}}_{r,w}$.

So $c_r(\mathring{\mathcal{B}}_{v,w} \cap rU^{-}\cdot B^{+}) \subseteq \mathring{\mathcal{B}}_{v,r} \times \mathring{\mathcal{B}}_{r,w}$

However,

$$\begin{array}{ccc} r\dot{u}^{-}\cdot B^{+} & = & \coprod_{v,w} \mathring{B}_{v,w} \cap r\dot{u}^{-}B^{+} \\ \simeq \downarrow & & \quad\quad \int \quad (\Leftarrow \text{ surjective for each } v,w) \\ \mathring{B}_{r}\times \mathring{B}^{r} & = & \coprod \mathring{B}_{v,r}\times \mathring{B}_{r,w} \end{array}$$

As a summary, we have

$$c_r : \mathring{B}_{v,w} \cap r\dot{u}^{-}\cdot B^{+} \simeq \mathring{B}_{v,r} \times \mathring{B}_{r,w}$$

As a consequence, $\mathring{B}_{v,w} \cap r\dot{u}^{-}\cdot B^{+} \neq \phi$

$\Leftrightarrow \mathring{B}_{v,r} \neq \phi$ and $\mathring{B}_{r,w} \neq \phi$

$\Leftrightarrow v \leq r \leq w$

<u>Lem.</u> Let $v \leq r \leq w$. Let Y be a connected component of $\mathring{B}_{v,w}(\mathbb{R})$. If $Y \subseteq r\dot{u}^{-}\cdot B^{+}$, then

1) $\bar{Y} \cap \mathring{B}_{v,r} = c_{r,+}(Y)$

2) $\bar{Y} \cap \mathring{B}_{r,w} = c_{r,-}(Y)$

3) $c_{r,+}(Y)$ is a connected component of $\mathring{B}_{v,r}(\mathbb{R})$

4) $c_{r,-}(Y)$ is a connected component of $\mathring{B}_{r,w}(\mathbb{R})$

Pf. Step 1: $r\cdot B^+ \in \overline{Y}$.

This is because

1. $\mathring{B}_{v,w}(\mathbb{R})$ is stable under the action of the torus $T(\mathbb{R})$

2. Any connected component of $\mathring{B}_{v,w}(\mathbb{R})$ is stable under $T_{>0}$

3. for any $p \in r u^- \cdot B^+$, $\overline{T_{>0}\cdot p} \ni r B^+$

Step 2

$$\begin{tikzcd}
\mathring{B}_{v,w}(\mathbb{R}) \cap \dot{r}\, U^- \cdot B^+ \arrow[r, "\sim"] \arrow[d, phantom, "\cup" , "\text{conn. comp}"'] & \mathring{B}_{v,r}(\mathbb{R}) \times \mathring{B}_{r,w}(\mathbb{R}) \arrow[d, phantom, "\cup", "\text{conn. comp}"'] \\
Y \arrow[r, "\sim"] & C_{r,+}(Y) \times C_{r,-}(Y)
\end{tikzcd}$$

Step 3.

$$\begin{tikzcd}
B_{v,w}(\mathbb{R}) \cap \dot{r}\, U^- \cdot B^+ \arrow[r, "\sim"] \arrow[d, phantom, "\cup"] & \left(\overline{B^- v \cdot B^+} \cap \mathring{B}_r(\mathbb{R})\right) \times \left(\overline{B^+ w B^+} \cap \mathring{B}^r(\mathbb{R})\right) \\
\overline{Y} \arrow[r] & (\overline{Y} \cap B_{v,r}) \times (\overline{Y} \cap B_{r,w})
\end{tikzcd}$$

$\overline{Y} \ni \dot{r} B^+$, $\quad \overline{Y} \cap B_{v,r} \ni \dot{r}\cdot B^+$, $\quad \overline{Y} \cap B_{r,w} \ni \dot{r} B^+$