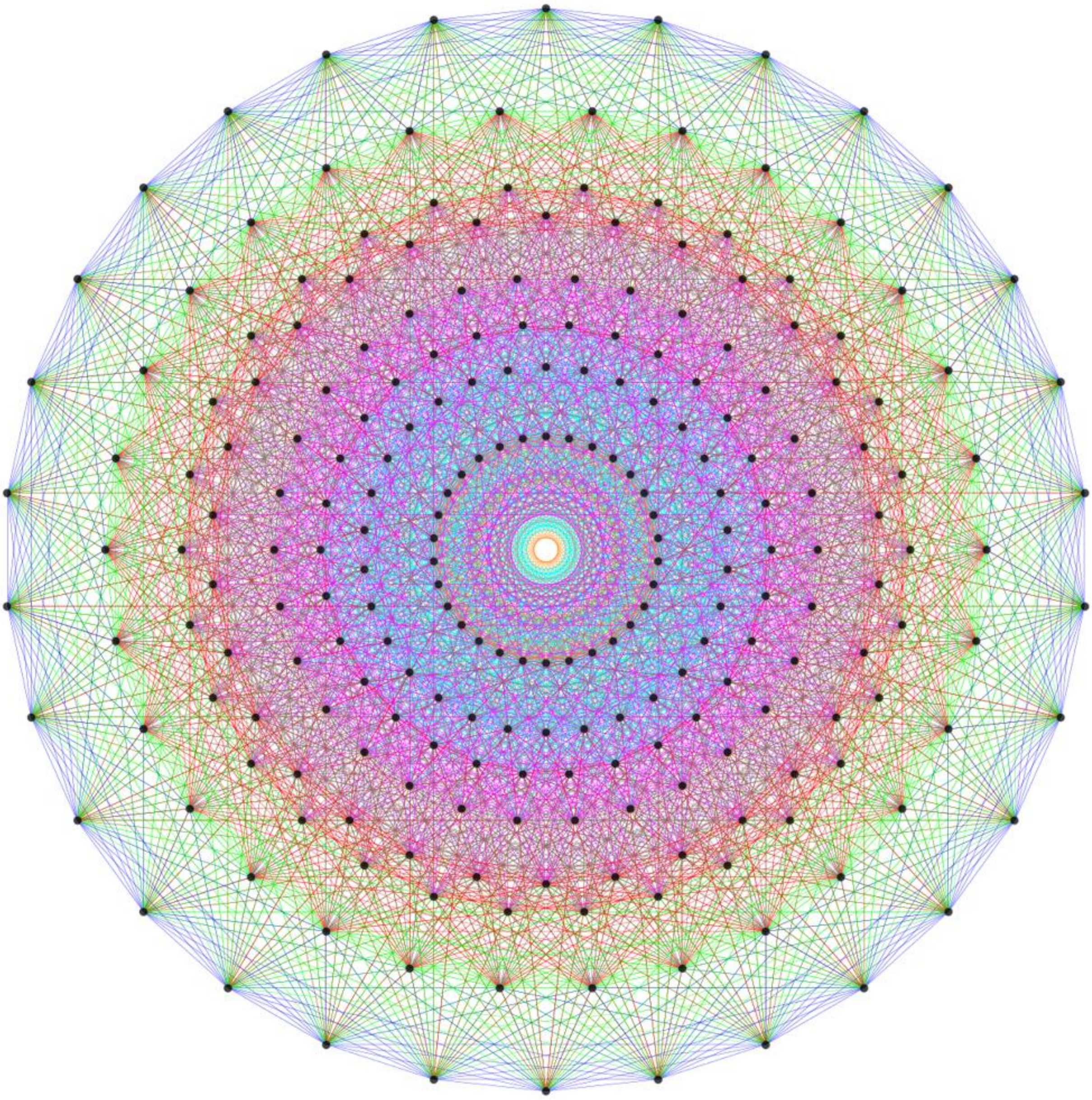

MATH6032 - Topics in Algebra II - 2021/22

Total positivity - Lecture 11

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Today we discuss the total positivity on flag manifolds.

Assume that G is a reductive group. Fix a pinning $(G, B^+, B^-, T = B^+ \cap B^-, x_i, y_i)$. Let $G_{\geq 0} = \langle x_i(a), y_i(a), t_i(a) \rangle_{i \in I, a > 0}$ be the totally nonnegative submonoid of G. We have the decomposition into cells

$$G_{\geq 0} = \coprod_{w_1, w_2 \in W} G_{w_1, w_2, >0}$$

where

$$G_{w_1, w_2, >0} \cong \mathbb{R}_{>0}^{l(w_1)+l(w_2)+\operatorname{rank} G}$$

and the closure relation

$$\overline{G_{w_1, w_2, >0}} = \coprod_{w_1' \leq w_1, w_2' \leq w_2} G_{w_1', w_2', >0}$$

Let $\mathcal{B} = G/B^+$ be the (full) flag variety. And $\mathcal{P}_K = G/P_K^+$ be the notation for the partial flag variety.

Definition 1

The totally positive flags are

$$\mathcal{B}_{>0} = U_{>0}^- \cdot B^+$$

and the totally nonegative flags $\mathcal{B}_{\geq 0}$ are the closure of $\mathcal{B}_{>0}$. ♣

In general, if G is a Kac-moody group, then $U_{>0}$ does not make sense. $\mathcal{B}_{\geq 0}$ is defined to be the closure of $U_{\geq 0}^- \cdot B^+$ in $\mathcal{B}$.

Note that in a reductive group G, $U_{\geq 0}^-$ is the closure of $U_{>0}^-$. So $U_{\geq 0}^- \cdot B^+$ is contained in the closure of $U_{>0}^- \cdot B^+$. So when G is a reductive group, the two definitions of $\mathcal{B}_{\geq 0}$ coincide.

<u>More on Kac-Moody groups</u>

If G is a reductive group, the Weyl group W has a longest element w_0 and $U_{>0}^-$ is defined to be $U_{w_0, >0}^-$. In general , W is an infinite group and there is no longest element. So $U_{>0}^-$ can not be defined. Another way is to use representation theory (for simply laced group via canonical basis).

Let G be a reductive group, V_λ be a highest weight representation, and $v_\lambda \in \beta$ be the canonical basis. For $u \in U^-$, write $u \cdot v_\lambda = \sum_{b \in \beta} c_b \in V_\lambda$. Here $c_b \in \mathbb{C}$.

Fact $u \in U_{>0}^- \Leftrightarrow c_b > 0, \forall b \in \beta$.

However, if G is a Kac-Moody group, $u \in U_{\geq 0}^-$, then there are only finitely many $b \in \beta, s.t. c_b \neq 0$, as u is a finite product of $y_i(>0)$. In particular, one never reach the lowest weight vector.

Example 1 *Let $G = \mathrm{GL}_2, \mathcal{B} = G/B^+ \cong \mathbb{P}^1$. B^+ corresponds to the point $[1:0] \in \mathbb{P}^1$. So*

$$y_i(a) \cdot B^+ = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix}$$

$\mathcal{B}_{>0} = \{[1:a], 0 < a < \infty\}$ is an open half circle in $\mathbb{RP}^1$ and $\mathcal{B}_{\geq 0} = \{[1:a], 0 \leq a \leq \infty\}$ is the closed half circle.

And a cell decomposition is given

$$\mathcal{B}_{>0} = \mathbb{R}_{>0}$$

$$\mathcal{B}_{\geq 0} = \mathbb{R}_{>0} \coprod \{0\} \coprod \{\infty\}$$

Also $x_i(a)\cdot B^- = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ 1 \end{bmatrix}$. *Hence we have a duality* $U^-_{>0}\cdot B^+ = u^+_{>0}\cdot B^-$.

General theory

Let G be a Kac-Moody group. Recall that we have the Bruhat decomposition

$$G = \coprod_{w\in W} B^+wB^+$$

.

Then $\mathcal{B} = \coprod_{w\in W} \mathring{\mathcal{B}}_w$ where $\mathring{\mathcal{B}}_w = B^+wB^+/B^+$ is a schubert cell. And schubert variety $\mathcal{B}_w :=$ Zaraski closure of $\mathring{\mathcal{B}}_w$.

we have $\mathcal{B}_w = \coprod_{w'\leq w} \mathring{\mathcal{B}}_{w'}$.

In particular $\mathcal{B} = \coprod_{w\in W} \mathring{\mathcal{B}}_w$ is a cellular decomposition of $\mathcal{B}$ and $\mathring{\mathcal{B}}_w \cong \mathbb{C}^{l(w)}$.

Back to $G = \mathrm{GL}_2$ case, $W = S_2 = \{1, s\}$. $\mathring{\mathcal{B}}_s \simeq \mathbb{C}$. $\mathring{\mathcal{B}}_1 \simeq$ pt.

Also $\mathcal{B} = \coprod_{u\in W} \mathring{\mathcal{B}}^u$ where $\mathring{\mathcal{B}}^u = B^-uB^+/B^+$ is of codimension $l(u)$. Let $\mathcal{B}^u$ be the Zaraski closure of $\mathring{\mathcal{B}}^u$. We have the **Birkhoff decomposition** $\mathcal{B}^u = \coprod_{u'\geq u} \mathring{\mathcal{B}}^{u'}$.

Remark In the special case when G is a reductive group. We have $B^- = \dot{w}_0 B^+ \dot{w}_0{}^{-1} = \dot{w}_0 B^+ \dot{w}_0$, so $\mathring{\mathcal{B}}^u = B^-uB^+/B^+ = \dot{w}_0 B^+ \dot{w}_0 u B^+/B^+ = \dot{w}_0 \mathring{\mathcal{B}}_{w_0 u}$.

Here $\dim \mathcal{B} = l(w_0)$, $\dim \mathring{\mathcal{B}}_{w_0 u} = l(w_0) - l(u)$. So $\mathring{\mathcal{B}}^u$ is of codim $l(u)$.

$u' \geq u \Leftrightarrow w_0 u' \leq w_0 u$ implies that the closure relation on $\mathring{\mathcal{B}}^u$ and $\mathring{\mathcal{B}}_{w_0 u}$ are decided by each other.

Definition 2

The ***open Richardson variety***

$$\mathring{\mathcal{B}}_{u,w} = \mathring{\mathcal{B}}^u \cap \mathring{\mathcal{B}}_w$$

And ***the closed Richardson variety***

$$\mathcal{B}_{u,w} = \mathcal{B}^u \cap \mathcal{B}_w$$

♣

Remark In the cohomology ring $H^*(B)$, we have the basis given by $[\mathcal{B}^{w_0 w}] = [\mathcal{B}_w]$. And the multiplication is given by

$$[\mathcal{B}^u] \cup [\mathcal{B}_w] = [\mathcal{B}_{u,w}].$$

Here the key feature of Richardson variety is that it is the intersection of B^+-orbits with B^--orbits. (and $\mathrm{Lie}(B^+) + \mathrm{Lie}(B^-) = \mathrm{Lie}(G)$) and such intersection is a transversal intersection

Proposition 1

Let $u, w \in W$. *The following are equivalent: (1)* $\mathring{\mathcal{B}}_{u,w} \neq \varnothing$; *(2)* $\mathcal{B}_{u,w} \neq \varnothing$; *(3)* $u \leqslant w$.

♠

Proof (1)$\Rightarrow$ (2): Obvious since $\mathcal{B}_{u,w} \supset \mathring{\mathcal{B}}_{u,w}$.

(2)$\Rightarrow$ (3):$\mathcal{B}_{u,w}$ is closed in $\mathbb{B}$ as $\mathcal{B}^u$ and $\mathcal{B}_w$ is closed. Also $\mathcal{B}_{u,w}$ is stable under the action of T as $\mathcal{B}^u$ and $\mathcal{B}_w$ is. If $\mathcal{B}_{u,w} \neq \varnothing$, then it contains a T-fixed point (limit of T^∞ of a point). The T-fixed point in $\mathcal{B}$ are $\{v\cdot B^+; w \in W\}$/(This can be proved from Bruhat decomposition).

[Passing to T-fixed point is a common trick in geometric representation theory as T-fixed points are often discrete and admit combinatorial description].

Suppose $v\cdot B^+ \in \mathring{\mathcal{B}}_{u,w}$ is a fixed point. Then $v\cdot B^+ \in \mathcal{B}_w$ so $\mathring{\mathcal{B}}_v = B^+v\cdot B^+/B^+ \subseteq \mathcal{B}_w$ and $v \leq w$.

Also $v\cdot B^+ \in \mathcal{B}^u$, so $\mathring{\mathcal{B}}^v = B^-v\cdot B^+/B^+ \subseteq \mathcal{B}^u$ and $v \geq w$. Therefore, $u \leq v \leq w$ and $u \leq w$.

(3)$\Rightarrow$ (1): There is a simple proof by studying the root subgroups, e.g.Kumar's book[2]. We won't follow this proof. Instead, we will give a structural description of $\mathring{\mathcal{B}}_{u,w}$ when $u \leq w$. We will follow [1] and [3]. ■

Before that, we come back to the $G=\mathrm{GL}_2$ case. Here $W=S_2$, $\{(v,w); v\le w\}=\{(1,s),(1,1),(s,s)\}$.

$$\mathring{\mathcal{B}}_{1,1}=\mathring{\mathcal{B}}^1\cap\mathring{\mathcal{B}}_1=B^+=[1:0]\in\mathbb{P}^1$$

$$\mathring{\mathcal{B}}_{s,s}=\mathring{\mathcal{B}}^s\cap\mathring{\mathcal{B}}_s=sB^+=[0:1]\in\mathbb{P}^1$$

$$\mathring{\mathcal{B}}_{1,s}=\mathring{\mathcal{B}}^1\cap\mathring{\mathcal{B}}_s=U^-\cdot B^+\cap B^+sB^+=\{y(a)\cdot B^+ : a\neq 0\}$$

Deodhar decomposition

Let $\underline{w}=s_{i_1}\cdots s_{i_n}$ be a reduced expression for w. Then $u\le w\Leftrightarrow\exists$ a subexpression $\underline{u}=t_{i_1}\cdots t_{i_n}$, where $t_i\in\{1,s_i\}$. But this subexpression is not unique in general.

Deodhar's idea: Fix a reduced expression: $\underline{w}=s_{i_1}\cdots s_{i_n}$, for any point in $\mathring{\mathcal{B}}_{u,w}$ we obtain a certain subexpression for u. This leads to a decomposition of $\mathring{\mathcal{B}}_{u,w}$.

Recall we have an isomorphism (case $u=1$)

$$\begin{aligned} y_{i_1}(\mathbb{R})\times\cdots\times y_{i_n}(\mathbb{R}) &\longrightarrow \mathring{\mathcal{B}}_{1,w}(\mathbb{R})\\ (y_1(a_1),\cdots,y_{i_n}(a_n)) &\longrightarrow p \end{aligned}$$

Here, from the point p, we not only get the element $(1,w)$, but we get the sequence $(a_1,\cdots,a_n)$. This is not the Deodhar's construction, but it illustrates the idea.

Consider $\mathcal{B}\times\mathcal{B}$ with the diagonal action of G. Then we have

$$G\backslash\left(G/B^+\times G/B^+\right)\longleftrightarrow B^+\backslash G/B^+\longleftrightarrow W.$$

Definition 3 (Relative position (a reformulation of the Bruhat decomposition))

We write $B_1\xrightarrow{w}B_2$ if (B_1,B_2) is in the G-orbit of $(B^+,w\cdot B^+)$. (B_1,B_2) is in a relative position w.r.t w. ♣

If $w=vv'$ with $l(w)=l(v)+l(v')$ then we have an isomorphism.

$$B^+vB^+\times^{B^+}B^+v'B^+\cong B^+wB^+$$

In other words, for any B_1,B_2 with $B_1\xrightarrow{w}B_2$, $\exists! B_3, s.t. B_1\xrightarrow{v}B_3\xrightarrow{v'}B_2$.

Particularly, $B\in\mathring{\mathcal{B}}_w$, where we have $B^+\xrightarrow{w}B$.

Definition 4 (Reduction map)

We set $\pi_v^w(B)$ be the unique element with $B^+\xrightarrow{v}\pi_v^w(B)\xrightarrow{v'}B$. π_v^w is called the reduction map. ♣

Now let $\underline{w}=s_{i_1}\cdots s_{i_n}$, set $\underline{w}_{(k)}=s_{i_1}\cdots s_{i_k}$. For any subexpression $\underline{v}=t_{i_1}\cdots t_{i_n}$, set $\underline{v}_{(k)}=t_{i_1}\cdots t_{i_k}$.

Definition 5 (Deodhar Component)

$$\mathring{\mathcal{B}}_{\underline{v},\underline{w}}=\left\{B\in\mathring{\mathcal{B}}_{v,w};\pi^w_{w_{(k)}}(B)\in B^-v_{(k)}\cdot B^+\quad\forall k\right\}$$

♣

By definition, for any fixed reduced expression $\underline{w}$.

$$\mathring{\mathcal{B}}_{v,w} = \coprod_{\underline{v} \text{ subexpression of } v \text{ in } \underline{w}} \mathring{\mathcal{B}}_{\underline{v},\underline{w}}$$

This is called Deodhar decomposition.

Remark Deodhar's motivation is a geometric interpretation of Kazhdan-Lusztig's R-polynomial, i.e.

$$R_{u,w}(q) = \#\mathring{\mathcal{B}}_{v,w}(\mathbb{F}_q) = \sum_{\text{Deodhar decomposition}} (q-1)^* q^{**}$$

$*$ means to some power. From the following theorem you will see that the above formula holds for abitrary field K and so $* = J^0_{\underline{u}}$ and $** = J^-_{\underline{u}}$.

Theorem 1 (Deodhar)

(1)$\mathring{\mathcal{B}}_{\underline{v},\underline{w}} \neq \varnothing$ iff $\underline{u}$ is a distinguished expression of $\underline{w}$.
(2) If $\underline{u}$ is a distinguished subexpression of $\underline{w}$, then $\mathring{\mathcal{B}}_{\underline{v},\underline{w}} \simeq (K^\times)^{\# J^0_{\underline{u}}} \times K^{\# J^-_{\underline{u}}}$, where $J^0_{\underline{u}}, J^-_{\underline{u}}$ are certain subsets of $\{1, 2, \cdots, n\}$

♡

Theorem 2 (Marsh-Rietsch)

(1)$\mathring{\mathcal{B}}_{\underline{v},\underline{w}} \cap \mathcal{B}_{\geq 0} \neq \varnothing$ iff $\underline{u}$ is a positive subexpression of $\underline{w}$.
(2)If $\underline{u}$ is a positive subexpresion of $\underline{w}$, then

$$\mathring{\mathcal{B}}_{\underline{v},\underline{w}} \cap \mathcal{B}_{\geq 0} \cong (\mathbb{R}_{>0})^{l(w)-l(u)}$$

as cells. ($|J^\circ_{\underline{u}}| = l(w) - l(u)$.)

♡

We will define distinguished expression and positive subexpression as follows

Definition 6

Set $J^{+/0/-}_{\underline{v}} = \{k : v_{(k-1)} < / = / > v_{(k)}\}$.
*We say that $\underline{v}$ is **distinguished** in $\underline{w}$ if*

$$v_{(k)} \leq v_{(k-1)} s_{i_k}, \forall k$$

i.e.,if $v_{(k-1)} \cdot s_{i_k} < v_{(k-1)}$ then $t_{i_k} = s_{i_k}$. (When it may go down, it will go down).
*We say that $\underline{v}$ is **positive** in $\underline{w}$ if*

$$v_{(k-1)} < v_{(k-1)} s_{i_k}, \forall k$$

(it is distinguished, and never goes down, as $v_{(k)} \in \{v_{(k-1)}, v_{(k-1)} s_{i_k}\} \geq v_{(k-1)}$).

♣

Example 2 $G = \mathrm{GL}_4, W = S_4 =< (12), (23), (34) >=< s_1, s_2, s_3 >$.

$$w = w_0 = s_3 s_2 s_1 s_3 s_2 s_3, v = s_2 s_3$$

The subexpressions of v are

1. $1111 s_2 s_3$
2. $1 s_2 111 s_3$
3. $1 s_2 1 s_3 11$
4. $s_3 s_2 1 s_3 s_2 1$

Then 1. is distinguished and positive, 2.,3.&4. are not distinguished.

Bibliography

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[3] R Marsh and Konstanze Rietsch. “Parametrizations of flag varieties”. In: *Representation Theory of the American Mathematical Society* 8.9 (2004), pp. 212–242.