THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 3
Due Date: 26th September 2019

Compulsory part

1. Show that A_n is a normal subgroup of S_n and compute S_n/A_n; that is, find a known group to which S_n/A_n is isomorphic.

2. A torsion group is a group all of whose elements have finite order. A group is torsion free if the identity is the only element of finite order. Prove that the torsion subgroup T of an abelian G is a normal subgroup of G, and that G/T is torsion free.

3. Let H be a normal subgroup of a group G, and let $m = (G : H)$. Show that $a^m \in H$ for every $a \in G$.

4. Let G be a group containing at least one subgroup of fixed finite order S. Show that the intersection of all subgroups of G of order s is a normal subgroup of G. [Hint: Use the fact that if H has order s, then so does $x^{-1}Hx$ for all $x \in G$.]

5. Show that the set of all $g \in G$ such that $i_g : G \to G$ is the identity inner automorphism i_e is a normal subgroup of a group G.

6. Using the properties $\det(AB) = \det(A)\det(B)$ and $\det(I_n) = 1$ for $n \times n$ matrices to show the following:

   (a) The $n \times n$ matrices with determinant 1 form a normal subgroup of $GL(n, \mathbb{R})$.

   (b) The $n \times n$ matrices with determinant ± 1 form a normal subgroup of $GL(n, \mathbb{R})$.

Optional part

1. Given any set S of a group G, show that it makes sense to speak of the smallest normal subgroup that contains S.

2. Let G be a group, and let $P(G)$ be the set of all subsets of G. For any $A, B \in P(G)$, let us define the product subset $AB = \{ab | a \in A, b \in B\}$.

   (a) Show that this multiplication is associative and has an identity element, but that $P(G)$ is not a group under this operation.

   (b) Show that if N is a normal subgroup of G, then the set of cosets of N is closed under the above operation on $P(G)$, and that this operation agrees withthe multiplication given by the formula in Corollary 14.5 of textbook.

   (c) Show (without using Corollary 14.5 of textbook) that the cosets of N in G form a group under the above operation. Is its identity element the same as the identity element of $P(G)$.