MATH3030 EXAM I–10/17/19 OUTLINED SOLUTION

- Print your name and student ID on the front page
- Give adequate explanation and justification for all your calculations and observations, and write your proofs in a clear and rigorous way
- Answer all questions.

(1) Let $\mathbb{Z}_{18}^{\times}$ be the group of all positive integers less than 18 and relative prime to 18, with the group operation given by the multiplication modulo 18.

(1) (10 points) List all the elements of $\mathbb{Z}_{18}^{\times}$.
$\mathbb{Z}_{18}^{\times} = \{1, 5, 7, 11, 13, 17\}$

(2) (10 points) Show that $\mathbb{Z}_{18}^{\times}$ is cyclic.
The order of 5 is 6 which is the order of $|\mathbb{Z}_{18}^{\times}|$, as $5^2 \equiv 7 \pmod{18}$ and $5^3 \equiv 17 \pmod{18}$.

(2) Let $G = S_3$, the symmetry group of $\{1, 2, 3\}$ and $G' = GL(2, \mathbb{Z}_2)$, the group of 2×2-matrices with nonzero determinant and with entries from $\mathbb{Z}_2$.

(1) (10 points) List the elements of S_3. What are the orders of these elements?
$S_3 = \{Id, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$. Id is of order 1. $(1,2)$, $(1,3)$ and $(2,3)$ are of order 2. $(1,2,3)$ and $(1,3,2)$ are of order 3.

(2) (10 points) List the elements of $GL(2, \mathbb{Z}_2)$. What are the orders of these elements?
$GL(2, \mathbb{Z}_2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is of order 1. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are of order 2. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ are of order 3.

(3) (10 points) Construct a group isomorphism from G to G' by specifying the image of every element of G.
Let $\phi : S_3 \to GL(2, \mathbb{Z}_2)$ be a map defined by $\phi(Id) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\phi(1,2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\phi(1,3) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\phi(2,3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\phi(1,2,3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and

$\phi(1,3,2) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. This is a group isomorphism.

(3) (1) (10 points) State the fundamental theorem of finite abelian groups
Book-work

(2) (15 points) Let G be a finite abelian group. Suppose that G has exactly 3 subgroups: $\{e\}$, G itself and another subgroup. Show that $G \cong \mathbb{Z}_{p^2}$ for some prime p.
Let n be the order of G. By the fundamental theorem of finite abelian groups, $G \cong \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_t}$ where $q_1, \ldots, q_t$ are powers of (not necessarily distinct) prime numbers.

We first claim that n is a prime power, that is, $n = p^r$ for some prime p and some positive integer r. Suppose not, there are two distinct primes r and s such that $r|n$ and $s|n$. Then G has at least 4 subgroups: $\{e\}$, G, a subgroup isomorphic to $\mathbb{Z}_r$ and a subgroup isomorphic to $\mathbb{Z}_s$. This follows plainly.

Next we would like to show that $n = p^2$. If $n = p$, then $G \cong \mathbb{Z}_p$ has only 2 subgroups. If $n = p^r$ with $r \geq 3$, then $G \cong \mathbb{Z}_{p^r}$ has $r+1$ subgroups. Therefore, $G \cong \mathbb{Z}_{p^2}$.

(4) (1) (15 points) Prove that every finite group has a composition series.
Note that because every finite group is a finite set, every chain of proper normal subgroups of a finite group has a maximal element and thus every finite group has a proper maximal normal subgroup.
Let G be a finite group. We proceed to the statement by mathematical induction on the order $|G| = n$.
When $n = 1$, it is trivial.
Suppose that any finite group having order less than n has a composition series. Let G be a group of order n.
If G is simple, then we are done.
If G is not simple, then it has a proper maximal normal subgroup H. Thus the quotient group G/H is simple. Applying the induction hypothesis on $|H|$ as $|H| < |G|$, H has a composition series, i.e.

$$\{e\} = N_m \lhd N_{m-1} \lhd \cdots \lhd N_1 = H.$$

Thus the series

$$\{e\} = N_m \lhd N_{m-1} \lhd \cdots \lhd N_1 = H \lhd N_0 = G$$

has a simple factor N_i/N_{i+1}, hence it is a composition series for G.

(2) (10 points) Find an example of two nonisomorphic groups with the same composition factors
Consider D_p and Z_{2p} for any odd prime p.