THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 8 Solution
Due Date: 14 November 2019

Compulsory part

1. Note that $D[x]$ is also a commutative ring with unity as D is an integral domain. To show that $D[x]$ is an integral domain, it suffices to show that $D[x]$ has no divisors of zero. Let $f(x) = a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ and $g(x) = b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0$ be two polynomials in $D[x]$ with a_n and b_m both nonzero. Because D is an integral domain, we know that a_nb_m never be zero, so the product $f(x)g(x)$ cannot be zero because the highest power term has nonzero coefficient a_nb_m.

2. (a) Observe that the degree of product of two polynomials is the sum of their degrees. The units in $D[x]$ are the units in D.
   (b) 1 and -1
   (c) 1, 2, 3, 4, 5 and 6.

3. (a) $f(x) = a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ and $g(x) = b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0$ be two polynomials in $F[x]$ with a_n and b_m both nonzero. By direct checking one has $D(f+g) = D(f)+D(g)$. It shows that D is a group homomorphism. But D is not a ring homomorphism as
   $$D(x\cdot x) = D(x^2) = 2x \neq 1 = 1\cdot 1 = D(x)\cdot D(x).$$
   (b) The kernel of D is F.
   (c) The image of $F[x]$ under D is $F[x]$ itself.

4. (a) Consider the map $g : F[x] \to F^F$ where $g(f(x))$ is the function $\phi \in F^F$ such that $\phi(a) = f(a)$ for all $a \in F$. It is not difficult to see that g is a ring homomorphism and it image is P_f. So P_f is a subring of F^F.
   (b) Let F be the finite field $\mathbb{Z}_2$. A function in $\mathbb{Z}_2^{\mathbb{Z}_2}$ has just two elements in both its domain and range. Thus there are only $2^2 = 4$ such functions in all. However, $\mathbb{Z}_2[x]$ is an infinite set, so it isn't isomorphic to $P_{\mathbb{Z}_2}$.

5. For $p = 2$, $x^2 + a$ has two possibilities $x^2 + 1$ and x^2 for which both of them are not irreducible since $x^2 + 1$ has zero 1 and x^2 has zero 0. For odd prime p, $x^p + a$ is not irreducible as
   $$(-a)^p + a = -a + a = 0.$$

6. (a) Let $f(x) = \sum_{i=0}^{\infty} a_ix^i$ and $g(x) = \sum_{j=0}^{\infty} b_jx^j$. It is not difficult to check that
   $$\overline{\sigma_m}(f+g) = \overline{\sigma_m}(f) + \overline{\sigma_m}(g).$$

Also we have

$$\begin{aligned}
\overline{\sigma_m}(fg) &= \overline{\sigma_m}\left(\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\overline{\sigma_m}\left(a_i b_{n-i}\right)\right)x^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\overline{\sigma_m}\left(a_i\right)\overline{\sigma_m}\left(b_{n-i}\right)\right)x^n \\
&= \overline{\sigma_m}(f)\overline{\sigma_m}(g).
\end{aligned}$$

These suggest that it is a ring homomorphism.

Let $h(x) \in \mathbb{Z}_m[x]$. Then obviously (by abusing the notation) $h(x) \in \mathbb{Z}[x]$ and $\overline{\sigma_m}(h) = h$. This show that it is onto.

(b) Let $f(x) = g(x)h(x)$ for $g(x)$ and $h(x)$ are both integral polynomials with the degrees of both $g(x)$ and $h(x)$ less than the degree n of $f(x)$. Applying the homomorphism $\overline{\sigma_m}$, we see that

$$\overline{\sigma_m}(f(x)) = \overline{\sigma_m}(g(x))\overline{\sigma_m}(h(x))$$

forms a factorization of $\overline{\sigma_m}(f(x))$ into two polynomials of degree less than the degree n of $\overline{\sigma_m}(f(x))$, contrary to the hypothesis. Thus $f(x)$ is irreducible in $\mathbb{Z}[x]$, and hence in $\mathbb{Q}[x]$.

(c) Considering $m = 5$, we see that $\overline{\sigma_m}(x^3+17x+36) = x^3+2x+1$ which is irreducible over $\mathbb{Z}_5$. By Part(b), we conclude that $x^3 + 17x + 36$ is irreducible in $\mathbb{Q}[x]$