THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 8
Due Date: 14 November 2019

Compulsory part

1. Prove that D is an integral domain, then $D[x]$ is an integral domain.

2. Let D be an integral domain and x an indeterminate.
   (a) Describe the units in $D[x]$.
   (b) Find the units in $\mathbb{Z}[x]$.
   (c) Find the units in $\mathbb{Z}_7[x]$.

3. Let F be a field of characteristic 0 and let D be the formal polynomial differentiation map, so that
$$D\left(a_0 + a_1 + a_2x^2 + \cdots + a_nx^n\right) = a_1 + 2 \cdot a_2x + \cdots + n \cdot a_nx^{n-1}.$$
   (a) Show that $D : F[x] \to F[x]$ is a group homomorphism of $\langle F[x], +\rangle$ into itself. Is D a ring homomorphism?
   (b) Find the kernel of D.
   (c) Find the image of $F[x]$ under D.

4. Let R be a ring, and let R^R be the set of all functions mapping R into R. For $\phi, \psi \in R^R$, define the sum $\phi + \psi$ by
$$(\phi + \psi)(r) = \phi(r) + \psi(r)$$
and the product $\phi \cdot \psi$ by
$$(\phi \cdot \psi)(r) = \phi(r)\psi(r)$$
for $r \in R$. Note that $\cdot$ is *not* function composition. It is known that $\langle R^R, +, \cdot\rangle$ is a ring. Let F be a field. An element ϕ of F^F is a **polynomial function on** F, if there exists $f(x) \in F[x]$ such that $\phi(a) = f(a)$ for all $a \in F$.
   (a) Show that the set P_f of all polynomial functions on F forms a subring of F^F.
   (b) Show that the ring P_f is not necessarily isomorphic to $F[x]$. (Hint: Consider the finite field.)

5. Show that for p a prime, the polynomial $x^p + a \in \mathbb{Z}_p[x]$ is not irreducible for any $a \in \mathbb{Z}_p$.

6. Let $\sigma_m : \mathbb{Z} \to \mathbb{Z}_m$ be the natural homomorphism given by
$$\sigma_m(a) = (\text{the remainder of } a \text{ when divided by } m)$$
for $a \in \mathbb{Z}$.

(a) Show that $\overline{\sigma_m} : \mathbb{Z}[x] \to \mathbb{Z}_m[x]$ given by

$$\overline{\sigma_m}(a_0 + a_1 x + \cdots + a_n x^n) = \sigma_m(a_0) + \sigma_m(a_1)x + \cdots + \sigma_m(a_n)x^n$$

is a homomorphism of $\mathbb{Z}[x]$ onto $\mathbb{Z}_m[x]$.

(b) Show that if $f(x) \in \mathbb{Z}[x]$ and $\overline{\sigma_m}(f(x))$ both have degree n and $\overline{\sigma_m}(f(x))$ does not have factor in $\mathbb{Z}_m[x]$ into two polynomials of degree less than n, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(c) Use part (b) to show that $x^3 + 17x + 36$ is irreducible in $\mathbb{Q}[x]$. (Hint: Try a prime value of m that simplifies the coefficients.)