THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 4
Due Date: 3rd October 2019

Compulsory part

1. Let $\phi : G \to G'$ be a group homomorphism. Show that if $|G|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G|$.

2. Show that any group homomorphism $\phi : G \to G'$ where $|G|$ is a prime must either be the trivial homomorphism or a one-to-one map.

3. Let $\phi : G \to H$ be a group homomorphism. Show that $\phi[G]$ is abelian if and only if for all $x, y \in G$, we have $xyx^{-1}y^{-1} \in Ker(\phi)$.

4. Let G be a group. Let $h, k \in G$ and let $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ be defined by $\phi(m, n) = h^m k^n$. Give a necessary and sufficient condition, involving h and k, for ϕ to be a homomorphism. Prove your condition.

5. Let G be an abelian group. The elements of finite order in G form a subgroup group of G. This subgroup is called the torsion subgroup of G. Find a torsion subgroup T of the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers.

6. It is known that: Let G be an abelian group. Let H be a subset of G consisting of the identity e together with all the order 2 elements in G. This H is a subgroup.

   Find a counterexample to the above with hypothesis that G is not abelian.

7. Show that if G is nonabelian, then the factor group $G/Z(G)$ is not cyclic. [Hint: Show that if the factor group $G/Z(G)$ is cyclic, then G is abelian.]

8. Show that a nonabelian group G of order pq where p and q are primes has a trivial center.

Optional part

1. Let $\phi : G \to G'$ with kernel H and let $a \in G$. Prove the set equality $\{x \in G : \phi(x) = \phi(a)\} = Ha$.

2. Find a necessary and sufficient condition on G such that the map ϕ described in Question 4 is a homomorphism for all choices of $h, k \in G$.

3. Prove that A_n is simple for $n \geq 5$, following the steps and hints given.

   (a) Show A_n contains every 3-cycle if $n \geq 3$.

   (b) Show A_n is generated by 3-cycles for $n \geq 3$. [Hint: Note that $(a, b)(c, d) = (a, c, b)(a, c, d)$ and $(a, c)(a, b) = (a, b, c)$.]

(c) Let r and s be fixed elements of $\{1, 2, \cdots, n\}$ for $n \geq 3$. Show that A_n is generated by n "special" 3-cycles of the form (r, s, i) for $1 \leq i \leq n$. [Hint: Show every 3-cycle is the product of "special" 3-cycles by computing

$(r, s, i)^2$, $\quad (r, s, j)(r, s, i)^2$, $\quad (r, s, j)^2(r, s, i)$, and $(r, s, i)^2(r, s, k)(r, s, j)^2(r, s, i)$.

Observe that these product give all possible types of 3-cycles.]

(d) Let N be a normal subgroup of A_n for $n \geq 3$. Show that if N contains a 3-cycle, then $N = A_n$. [Hint: Show that $(r, s, i) \in N$ implies that $(r, s, j) \in N$ for $j = 1, 2, \cdots, n$ by computing

$$((r, s)(i, j))\,(r, s, i)^2\,((r, s)(i, j))^{-1}\,.$$

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(e) Let N be a nontrival normal subgroup of A_n for $n \geq 5$. Show that one of the following cases must hold, and conclude in each case that $N = A_n$.

Case I N contains a 3-cycle.

Case II N contains a product of disjoint cycles, at least one of which has length greater than 3. [Hint: Suppose N contains the disjoint product $\sigma = \mu(a_1, a_2, \cdots, a_r)$. Show $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$ is in N, and compute it.]

Case III N contains a disjoint product of the form $\sigma = \mu(a_4, a_5, a_6)(a_1, a_2, a_3)$. [Hint: Show $\sigma^{-1}(a_1, a_2, a_4)\sigma(a_1, a_2, a_4)^{-1}$ is in N, and compute it.]

Case IV N contains a disjoint product of the form $\sigma = \mu(a_1, a_2, a_3)$ where μ is a product of disjoint 2-cycles. [Hint: Show σ^2 is in N, and compute it.]

Case V N contains a disjoint product σ of the form $\sigma = \mu(a_3, a_4)(a_1, a_2)$, where μ is a product of of an even number of disjoint 2-cycles. [Hint: Show that $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$ is in N, and compute it to deduce that $\alpha = (a_2, a_4)(a_1, a_3)$ is in N. Using $n \geq 5$ for the first time, find $i \neq a_1, a_2, a_3, a_4 \in \{1, 2, 3, \cdots, n\}$. Let $\beta = (a_1, a_3, i)$. Show that $\beta^{-1}\alpha\beta\alpha \in N$, and compute it.]