THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 2
Due Date: 19th September 2019

Compulsory part

1. Show that a group with at least two elements but with no proper nontrivial subgroups must be finite and of prime order.

2. Show that if H is a subgroup of index 2 in a finite group G, then every right coset of H is also a left coset of H.

3. Let H and K be subgroups of a group G. Define $\sim$ on G by $a \sim b$ if and only if $a = hbk$ for some $h \in H$ and $k \in K$.

   (a) Prove that $\sim$ is an equivalence relation on G.

   (b) Describe the elements in the equivalence class containing $a \in G$. (These equivalence classes are called double cosets.)

4. Show that a finite cyclic group of order n has exactly one subgroup of each order d dividing n, and that these are all the subgroups it has.

5. The part of the decomposition of G in Theorem 11.12 (of the textbook) corresponding to the subgroups of prime-power order can also be written in the form $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$, where m_i divides m_{i+1} for $i = 1, 2, \cdots, r-1$. The numbers m_i can be shown to be unique and are the **torsion coefficients** of G.

   (a) Find the torsion coefficients of $\mathbb{Z}_4 \times \mathbb{Z}_9$.

   (b) Find the torsion coefficients of $\mathbb{Z}_6 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20}$.

   (c) Describe an algorithm to find the torsion coefficients of a direct product of cyclic groups.

6. Let G be an abelian group. Let H be the subset of G consisting of the identity e together with all elements of G of order 2. Show that H is a subgroup of G.

7. Let H and K be groups and let $G = H \times K$. Recall that both H and K appear as subgroups of G in a natural way. Show that these subgroups H (actually $H \times \{e\}$) and K (actually $\{e\} \times K$) have the following properties.

   (a) Every element in G is of the form hk for some $h \in H$ and $k \in K$.

   (b) $kh = hk$ for all $h \in H$ and $k \in K$.

   (c) $H \cap K = \{e\}$.

8. Let H and K be subgroups of a group G satisfying the three properties listed in the preceding exercise. Show that for each $g \in G$, the expression $g = hk$ for $h \in H$ and $k \in K$ is unique. Then let each g be renamed (h, k). Show that, under this renaming, G becomes structurally identical (isomorphic)to $H \times K$.

Optional Part

1. The **Euler phi-function** is defined for a positive integer n by $\varphi(n) = s$, where s is the number of positive integers less than or equal to n that are relatively prime to n. Use Question 4 to show that
$$n = \sum_{d|n} \varphi(d),$$
the sum being taken over all positive integers d dividing n. [Hint: Note that the number of generators of $\mathbb{Z}_d$ is $\varphi(d)$.]

2. Let G be a finite group. Show that if for each positive integer m the number of solutions x of the equation $x^m = e$ in G is at most m, then G is cyclic. [Hint: Use the previous exercise.]

3. Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

4. Let G, H and K be finitely generated abelian groups. Show that if $G \times K$ is isomorphic to $H \times K$, then G and H are isomorphic.