THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 1
Due Date: 12th September 2019

Compulsory part

1. Show that if G is a finite group with identity e and with an even number of elements, then there is $a \neq e$ in G such that $a * a = e$.

2. Show that every group G with identity e and such that $x * x = e$ for all $x \in G$ is abelian. [Hint: Consider $(a * b) * (a * b)$.]

3. Let a nonempty finite subset H of a group G be closed under the binary operation of G. Show that H is a subgroup of G.

4. For sets H and K, we define the intersection $H \cap K$ by

$$H \cap K = \{x | x \in H \text{ and } x \in K\}.$$

Show that if $H \leq G$ and $K \leq G$, then $H \cap K \leq G$. (Reminder: $\leq$ denotes "is a subgroup of," not "is a subset of.")

5. Show that a group that has only a finite number of subgroups must be a finite group.

6. Let p and q be distinct prime numbers. Find the number of generators of the cyclic group $\mathbb{Z}_{pq}$.

7. Show that S_n is a nonabelian group for $n \geq 3$.

8. If A is a set, then a subgroup H of S_A is transitive on A if for each $a, b \in A$ there exists $\sigma \in H$ such that $\sigma(a) = b$. Show that if A is nonempty finite set, then there exists a finite cyclic subgroup H of S_A with $|H| = |A|$ that is transitive on A.

9. Prove the following about S_n if $n \geq 3$.

   (a) Every permutation in S_n can be written as a product of at most $n - 1$ transpositions.

   (b) Every permutation in S_n that is not a cycle can be written as a product of at most $n - 2$ transpositions.

   (c) Every odd permutation in S_n can be written as a product of $2n + 3$ transpositions, and every even permutation in S_n as a product of $2n + 8$ transpositions

10. Show that for every subgroup H of S_n for $n \geq 2$, either all the permutations in H are even or exactly half of them are even.

Optional Part

1. Let $\langle G, \cdot \rangle$ be a group. Consider the binary operation $*$ on the set G defined by

$$a * b = b \cdot a$$

for $a, b \in G$. Show that $\langle G, * \rangle$ is a group and that $\langle G, * \rangle$ is isomorphic to $\langle G, \cdot \rangle$. [Hint: Consider the map ϕ with $\phi(a) = a'$ for $a \in G$ where a' is the inverse of a in the group $\langle G, \cdot \rangle$.]

2. Show that a group with no proper nontrivial subgroups is cyclic.

3. Let G be an abelian group and let H and K be finite cyclic subgroups with $|H| = r$ and $|K| = s$.

   (a) Show that if r and s are relatively prime, then G contains a cyclic subgroup of order rs.

   (b) Generalizing part (a), Show that G contains a cyclic subgroup of order the least common multiple of r and s.

4. A permutation matrix is one that can be obtained from an identity matrix by reordering its rows. If P is an $n \times n$ permutation matrix and A is any $n \times n$ matrix and $C = PA$, then C can be obtained from A by making precisely the same reordering of the rows of A as the reordering of the rows which produced P from I_n.

   (a) Show that every finite group of order n is isomorphic to a group consisting of $n \times n$ permutation matrix under matrix multiplication.

   (b) For each of the four elements e, a, b, and c in the Table 5.11 (in the textbook) for the group V, give a specific 4×4 matrix that corresponds to it under such an isomorphism.

5. Show that S_n is generated by $\{(1, 2), (1, 2, 3, \ldots, n)\}$. [Hint: Show that as r varies, $(1, 2, 3, \ldots, n)^r(1, 2)(1, 2, 3, \ldots, n)^{n-r}$ gives all the transpositions $(1, 2), (2, 3), (3, 4), \cdots, (n-1, n), (n, 1)$. Then show that any transposition is a product of some of these transpositions and use Corollary 9.12.]