THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Tutorial 6
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1. (a) How many different ways can the cyclic group C_3 of order three act on the set $\{1, 2, 3, 4\}$?

   (b) How many different ways can the cyclic group C_4 of order four act on the set $\{1, 2, 3\}$?

Solution. (a) Consider $G = C_3$ acting on $S = \{1, 2, 3, 4\}$. The action decomposes S into a disjoint union of orbits. By the Orbit-Stabiliser Theorem, $|Ga| = |(G : G_a)|$ divides $|G|$ for all $a \in S$. Hence the orbits have size 1 or 3. There are therefore two possibilities:

Case 1: S is the union of four orbits all of size 1. Then G fixes all elements of S and therefore there is precisely one action.

Case 2: S is the union of an orbit of size 3 and an orbit of size 1.
The orbit of size 1 is a fixed point. There are 4 choices for this element.
Now consider the orbit of size 3. Suppose it consists of the three elements $\alpha, \beta, \gamma \in S$. Writing $G = \langle x \rangle$ where x has order 3, then the Orbit-Stabiliser Theorem shows that $\{\alpha, \beta, \gamma\} = \{\alpha, x\alpha, x^2\alpha\}$. Hence there are two possibilities: either $x\alpha = \beta$ or $x\alpha = \gamma$. Thus having chosen the fixed point, there are two further choices for the action.

In conclusion, there are $1 + 4 \times 2 = 9$ different actions of C_3 on $\{1, 2, 3, 4\}$.

(b) Consider $G = C_4$ acting on $S = \{1, 2, 3\}$. The action decomposes S into a disjoint union of orbits. By the Orbit-Stabiliser Theorem, $|Ga| = |(G : G_a)|$ divides $|G|$ for all $a \in S$. Hence the orbits have size 1 or 2. There are therefore two possibilities:

Case 1: S is the union of four orbits all of size 1. Then G fixes all elements of S and therefore there is precisely one action.

Case 2: S is the union of an orbit of size 2 and an orbit of size 1.
The orbit of size 1 is a fixed point. There are 3 choices for this element.
Now consider the orbit of size 2. Suppose it consists of the two elements $\alpha, \beta \in S$. The Orbit-Stabiliser Theorem tells us that $|G_a| = 2$, so if $G = \langle x \rangle$ then $G_a = \langle x^2 \rangle$ is the unique subgroup of G of order 2. Then we have $x\alpha = \beta$ and $x\beta = \alpha$. Hence the action of G on $\{\alpha, \beta\}$ is already uniquely determined, and we describe the whole action by specifying which point in S is the fixed point.

In conclusion, there are $1 + 3 = 4$ different actions of C_4 on $\{1, 2, 3\}$.

◀

2. Let G be a group and let Γ and Δ be sets such that G acts on Γ and on Δ. Define

$$g(\gamma, \delta) = (g\gamma, g\delta)$$

for all $\gamma \in \Gamma, \delta \in \Delta$ and $g \in G$.

   (a) Verify that this is an action of G on the set $\Gamma \times \Delta$.

   (b) Verify that the stabiliser of the pair (γ, δ) in this action equals the intersection of the stabilisers of γ and δ.

   (c) If G acts transitively on the non-empty set S, show that

$$D := \{(w, w) | w \in S\}$$

   is an orbit of S on $S \times S$. Deduce that G acts transitively on $S \times S$ if and only if $|S| = 1$.

Solution. (a) Firstly

$$h[g(\gamma, \delta)] = h(g\gamma, g\delta) = (h(g\gamma), h(g\delta)) = (hg\gamma, hg\delta) = hg(\gamma, \delta)$$

for all $(\gamma, \delta) \in \Gamma \times \Delta$ and all $g, h \in G$. Secondly

$$e(\gamma, \delta) = (e\gamma, e\delta) = (\gamma, \delta),$$

for all $(\gamma, \delta) \in \Gamma \times \Delta$.

(b)

$$G_{(\gamma,\delta)} = \{g \in G | g(\gamma, \delta) = (\gamma, \delta)\} = \{g \in G | g\gamma = \gamma \text{ and } g\delta = \delta)\} = G_\gamma \cap G_\delta.$$

(c) Let $(w, w) \in D$. For any $(w', w') \in D$, we can find a $g \in G$ such that $g(w, w) = (w', w')$ as G acts transitively on S: $gw = w'$. This shows that D is an orbit.
If $|S| = 1$, then it is trivial. Conversely, if G acts transitively then the orbit D must equal $S \times S$. If $w, w' \in S$, then $(w, w') \in S \times S = D$, which forces that $w = w'$. Hence $|S| = 1$.

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3. (a) There is a natural action of S_n on $X = \{1, 2, \ldots, n\}$. For $n \geq 2$, How many orbits does S_n have on $X \times X$?

   (b) Repeat part (a) with the action of the alternating group A_n on $X = \{1, 2, \ldots, n\}$ for $n \geq 3$.

Solution. (a) Consider the action of S_n on $X \times X$. Then $D = \{(w, w) | w \in X\}$ is an orbit as S_n acts transitively on X. If $n = |X| = 1$, then $D = X \times X$ and there is just one orbit.

Suppose $n \geq 2$, so D is a proper subset of $X \times X$. Consider two pairs not in D, say (w_1, w_2) and (w_1', w_2'). Then $w_1 \neq w_2$ and $w_1' \neq w_2'$. Define a permutation σ of X by defining it to map w_1 to w_1' and to map w_2 to w_2' and then to map the remaining $n-2$ points in the domain bijectively to the remaining$n-2$ points in the codomain. Then $\sigma \in S_n$ and

$$\sigma(w_1, w_2) = (\sigma w_1, \sigma w_2) = (w_1', w_2').$$

So $(X \times X) \setminus D$ is an orbit, and S_n has two orbits on $X \times X$ when $n \geq 2$.

(b) Consider the action of A_n on $X \times X$. Then $D = \{(w, w) | w \in X\}$ is an orbit as A_n acts transitively on X.

At first, let's assume $n \geq 4$. We have constructed in part (a) a permutation σ above which maps (w_1, w_2) to (w_1', w_2'). If σ is even, then there is nothing more to do. If, however, σ is an odd permutation, now apply a transposition τ which swaps some pair of points, neither of which is w_1' or w_2'. (This is possible since $n \geq 4$.) Then $\sigma' := \sigma\tau$ is an even permutation and

$$\sigma'(w_1, w_2) = (\sigma' w_1, \sigma' w_2) = (\sigma\tau w_1', \sigma\tau w_2') = (\sigma w_1', \sigma w_2') = (w_1', w_2').$$

(since τ fixes bothw_1' and w_2'). Thus, if $n \geq 4$, A_n acts transitively on $(X \times X) \setminus D$, and so A_n has two orbits on $X \times X$ when $n \geq 4$.

Now consider the case $n = 3$. Then $|A_3| = 3$ and $(X \times X) \setminus D$ has 6 elements. No point in $X \times X$ is fixed under the action of A_3 and hence the orbits all have length 3 (by the Orbit-Stabiliser Theorem). Hence there are three orbits for A_3 on $X \times X$.

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4. Let F be a finite field with q elements. By considering the action of $GL(n, F)$ on $F^n - \{\mathbf{0}\}$, show that

$$|GL(n, F)| = \prod_{i=0}^{n-1} (q^n - q^i).$$

Solution.

- Consider the standard action of $GL(n, F)$ on F^n (i.e. matrix multiplication).
- Since nonsingular matrices do not take a nonzero vector to the zero vector, it follows that the action of $GL(n, F)$ on $F^n - \{\mathbf{0}\}$ is defined.
- Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in F^n$.

- Consider the formula
$$|G| = |G \cdot \mathbf{x}| \cdot |G_{\mathbf{x}}|$$
where $G \cdot \mathbf{x}$ is the orbit containing $\mathbf{x}$ and $G_{\mathbf{x}}$ is the stabiliser of $\mathbf{x}$.
- These are the sets we need to determine.

$G \cdot \mathbf{x}$:
  - We claim that $G \cdot \mathbf{x} = F^n - \{\mathbf{0}\}$ (i.e. the action is transitive).
  - Recall from Linear Algebra that every nonzero vector in a finite-dimensional vector space can be extended to a basis.
  - More precisely, if $\mathbf{v} \in V$ is nonzero and dim$V = n$, then there is a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of V such that $\mathbf{v}_1 = \mathbf{v}$.
  - Now back to our claim, suppose $\mathbf{v} \in F^n - \{\mathbf{0}\}$, then we can find a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of F^n such that $\mathbf{v}_1 = \mathbf{v}$.
  - Let $A = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n]$.
  - Then A is nonsingular because its columns are linearly independent.
  - What's more, $A\mathbf{x} = \mathbf{v}_1 = \mathbf{v}$.
  - This shows that the action is indeed transitive.
  - Hence $|G \cdot \mathbf{x}| = q^n - 1$.

$G_{\mathbf{x}}$:
  - Observe that for an $n \times n$ matrix A, $A\mathbf{x}$ is nothing but the first column of A.
  - The stabiliser $G_{\mathbf{x}}$ of $\mathbf{x}$ is the set consisting of all nonsingular $n \times n$ matrices A such that $A\mathbf{x} = \mathbf{x}$,
  - in other words, those A whose first column is equal to $\mathbf{x}$.
  - Taking also the fact that the matrices are nonsingular into account, we see that these matrices are of the form
$$\left[\begin{array}{c|ccc} 1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & \in GL(n-1,F) & \\ 0 & & & \end{array}\right],$$
  - where the $*$'s are arbitrary, and there are $n-1$ of them.
  - It follows that $|G_{\mathbf{x}}| = q^{n-1}|GL(n-1,F)|$.

- Now, by putting what we have computed into the formula above, we get
$$|GL(n,F)| = (q^n - 1) \cdot \left(q^{n-1}|GL(n-1,F)|\right).$$
- By induction, we have
$$\begin{aligned} |GL(n,F)| &= [q^{n-1}(q^n-1)][q^{n-2}(q^{n-1}-1)]\cdots[q(q^2-1)] \cdot |GL(1,F)| \\ &= [q^{n-1}(q^n-1)][q^{n-2}(q^{n-1}-1)]\cdots[q(q^2-1)](q-1) \\ &= \prod_{i=0}^{n-1}(q^n - q^i). \end{aligned}$$

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