THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Tutorial 1
Date: 12th September 2019

1. Find the sign of each of the following permutations:

   (a) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 2 & 1 \end{pmatrix}$

   (b) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 8 & 2 & 4 & 1 & 3 & 7 \end{pmatrix}$

Solution. (a) Noting that

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \end{pmatrix},$$

the sign is -1.

(b)

$$\begin{aligned} \sigma &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 8 & 2 & 4 & 1 & 3 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 5 & 4 & 2 & 6 \end{pmatrix}\begin{pmatrix} 3 & 8 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 5 \end{pmatrix}\begin{pmatrix} 3 & 7 \end{pmatrix}\begin{pmatrix} 3 & 8 \end{pmatrix}, \end{aligned}$$

the sign is 1.

◀

2. Find all the subgroups of S_3 and D_3.

Solution. Note that D_3 can be views as a subgroup of S_3 because D_3 can be considered as a group permuting the three vertices of an equilateral triangle. So D_3 is isomorphic to S_3 as they both have six elements. It suffices to find all the subgroups of S_3 only. By Lagrange's Theorem, the order of subgroup should be 1, 2, 3 or 6. First of all, subgroups of order 1 or 6 are $\{Id\}$ and S_3 respectively. For the subgroup of order 2 or 3, it found that it is cyclic by using the corollary of theorem of Lagrange as it is of prime order. We list those subgroups of order 2 or 3 below: $\langle(1\ \ 2)\rangle$, $\langle(1\ \ 3)\rangle$, $\langle(2\ \ 3)\rangle$, and $\langle(1\ \ 2\ \ 3)\rangle$. To conclude, all the subgroups of S_3 are $\{Id\}$, $\langle(1\ \ 2)\rangle$, $\langle(1\ \ 3)\rangle$, $\langle(2\ \ 3)\rangle$,$\langle(1\ \ 2\ \ 3)\rangle$ and S_3.

◀

3. List all the elements of S_4 according to its cycle patterns.

Solution. S_4 has 5 cycle patterns: (i) trivial element; (ii) cycles of length 2; (iii) products of two disjoint cycles of length 2; (iv) cycles of length 3; (v) cycles of length 4. A complete list of elements in S_4 (in cycle notation) is

Cycle length	Elements in S_4
(i)	Id
(ii)	$(1\ \ 2)\,,\ (1\ \ 3)\,,\ (1\ \ 4)\,,\ (2\ \ 3)\,,\ (2\ \ 4)\,,\ (3\ \ 4)$
(iii)	$(1\ \ 2)\,(3\ \ 4)\,,\ (1\ \ 3)\,(2\ \ 4)\,,\ (1\ \ 4)\,(2\ \ 3)$
(iv)	$(1\ \ 2\ \ 3)\,,\ (1\ \ 3\ \ 2)\,,\ (1\ \ 2\ \ 4)\,,\ (1\ \ 4\ \ 2)\,,$
	$(1\ \ 3\ \ 4)\,,\ (1\ \ 4\ \ 3)\,,\ (2\ \ 3\ \ 4)\,,\ (2\ \ 4\ \ 3)$
(v)	$(1\ \ 2\ \ 3\ \ 4)\,,\ (1\ \ 3\ \ 4\ \ 2)\,,\ (1\ \ 4\ \ 2\ \ 3)\,,$
	$(1\ \ 2\ \ 4\ \ 3)\,,\ (1\ \ 4\ \ 3\ \ 2)\,,\ (1\ \ 3\ \ 2\ \ 4)$

◀

4. Show that for $n \geq 3$ A_n is generated by all 3-cycles in S_n.

Solution. Let H be the subgroup generated by all 3-cycles in S_n. We wish to show that $H = A_n$.

$\subseteq$: All 3-cycle and its inverse are even and are 3-cycles.

$\supseteq$: Every element in A_n can be written as the product of an even number of 2-cycles. We then pair up the adjacent 2-cycles and thus it suffices to show that the product of any pair of 2-cycles can be written as the product of some 3-cycles. For distinct i, j, k, ℓ, we have the following three possibilities:

$$(i\ \ j)\,(i\ \ j) = Id,$$

$$(i\ \ j)\,(j\ \ k) = (i\ \ j\ \ k)\,,$$

and

$$(i\ \ j)\,(k\ \ \ell) = (i\ \ j\ \ k)\,(j\ \ k\ \ \ell)\,.$$

◀