MMAT 5340 Assignment #5

Please submit your assignment online on Blackboard

Due at 12 p.m. on Wednesday, October 20, 2021

1. Let $(\xi_k)_{k\geq 1}$ be a sequence of identically independent distributed random variables with standard Gaussian distribution, i.e. $\xi_k \sim \mathcal{N}(0,1)$. We define $X = (X_n)_{n\geq 0}$ as follows:

$$X_0 := 0,\ X_n := \sum_{k=1}^{n} \frac{1}{k}\xi_k,\ \text{ for all } n \geq 1.$$

   (a) Prove that X is a martingale.

   (b) Recall that $C := \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. Prove that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[|X_n|^2\right] < \infty.$$

   (c) By the convergence theorem of the martingale (Theorem 2.4), we know that $X_n \to X_\infty$ a.s. and in L^2 for some random variable X_∞ as $n \to \infty$.

      i. Compute the characteristic function ψ_n of X_n, where ψ_n is defined as

$$\psi_n(\theta) := \mathbb{E}\left[e^{i\theta X_n}\right],\ \theta \in \mathbb{R}\,.$$

      ii. Compute

$$\psi(\theta) := \lim_{n\to\infty} \psi_n(\theta),\ \theta \in \mathbb{R}\,.$$

      iii. Please identify the distribution of X_∞.
      (Hint: ψ is the characteristic function of X_∞ and the distribution of a random variable is uniquely determined by its characteristic function.)

2. Let $(\xi_k)_{k\geq 1}$ be a sequence of identically independent distributed random variables such that $\mathbb{P}[\xi_k = \pm 1] = \frac{1}{2}$. We define $X = (X_n)_{n\geq 0}$ as follows:

$$X_0 := 0,$$
$$X_n := \sum_{k=1}^{n} 2^{k-1}\xi_k \mathbb{1}_{\{k\leq\tau\}}, \text{ where } \tau := \inf\{k \in \mathbb{N} : \xi_k = 1\}.$$

(a) Prove that X is a martingale.

(b) Compute $\mathbb{P}[\tau > n]$ and deduce that $\mathbb{P}[\tau < +\infty] = 1$
(Hint: $\{\tau > n\} = \{\xi_1 = \cdots = \xi_n = -1\}$.)

(c) Prove that $X_\tau = 1$ a.s.
Conclusion: In the above example

$$1 = \mathbb{E}[X_\tau] \neq \mathbb{E}[X_0] = 0.$$

(d) Compute $\mathbb{E}[|X_n|]$ and prove that

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|] < \infty, \quad \text{and} \quad \lim_{n\to\infty} X_n = X_\tau \text{ a.s.}$$

Hint: $\mathbb{E}\left[|X_n|\right] = \mathbb{E}\left[|X_n|\mathbb{1}_{\{\tau>n\}}\right] + \mathbb{E}\left[|X_\tau|\mathbb{1}_{\{\tau\leq n\}}\right]$.