# MMAT 5340 Assignment #4

Please submit your assignment online on Blackboard

Due at 12 p.m.(noon) on Wednesday, October 13, 2021

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0,1,\cdots}$.

   (a) Let $\tau_1, \tau_2$ be two $\mathbb{F}$-stopping times. Prove that

   $$\tau_1 \wedge \tau_2 := \min(\tau_1, \tau_2)$$
   $$\tau_1 \vee \tau_2 := \max(\tau_1, \tau_2)$$

   are both stopping times.

   (b) Let $\tau$ be an $\mathbb{F}$-stopping time. Prove that $\tau+1$ is also an $\mathbb{F}$-stopping time.

2. Let $X_0 = 0, X_n = \sum_{k=1}^{n} \xi_k$, where $(\xi_k)_{k\geq 1}$ is a sequence of independent and identically distributed random variables such that $\mathbb{P}[\xi_k = \pm 1] = \frac{1}{2}$. Let $M$ and $N$ be two positive integers and define

   $$\tau := \min\{n \geq 0 : X_n = -N \text{ or } X_n = M\}.$$

   (a) Prove that $\tau$ is an $\mathbb{F}$-stopping time, where $\mathbb{F}$ is the natural filtration generated by $X$.

   (b) Assume that $\tau < +\infty$ a.s., prove that $\mathbb{P}[X_\tau \in \{-N, M\}] = 1$.

   (c) Under the condition of (b), compute $\mathbb{E}[X_\tau]$ and $\mathbb{P}[X_\tau = -N]$.
   Hint: Let $X$ be a martingale and $\tau$ be a stopping time with respect to a filtration $\mathbb{F}$, and if $\tau < \infty$ and the process $(X_{\tau\wedge n})_{n\geq 0}$ is uniformly bounded. Then $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.