MMAT 5340 Assignment #2

Please submit your assignment online on Blackboard

Due at 12 p.m. on Wednesday, September 22, 2021

1. Let X and Y be independent random variables, and
$$\mathbb{P}[X = i] = \mathbb{P}[Y = i] = \frac{1}{3}, \quad \text{for } i \in \{-1, 0, 1\}.$$
Compute the following values
   a) $\mathbb{E}[Y]$.
   b) $\mathbb{E}[X + Y|X]$.

2. Let X and Y be independent random variables with standard Gaussian distribution, i.e. $X, Y \sim \mathcal{N}(0,\, 1)$, then for $\rho \in [-1, 1]$,
$$Z := \sqrt{1-\rho^2}X + \rho Y.$$
   a) Show that $Cov(Z, Y) = \rho$.
   b) Show that $\mathbb{E}[Z|Y] = \rho Y$.

   Hint: Recall Lemma 1.18 in lecture note that given $\mathbb{E}[|X|], \mathbb{E}[|XY|] < \infty$, then $\mathbb{E}[X|Z] = \mathbb{E}[X]$ if X, Z are independent, $\mathbb{E}[XY|Z] = \mathbb{E}[X|Z]Y$ if Y is $\sigma(Z)-$measurable.

3. Let X and Y be independent continuous random variables with probability density function $\rho_x(x), \rho_y(y)$, respectively. Consider a bounded measurable function g, let
$$f(y) := \mathbb{E}[g(X + y)].$$
Show that $\mathbb{E}[g(X + Y)|Y] = f(Y)$.

   Hint: By definition of the conditional expectation, it is enough to show that $\mathbb{E}[f(Y) \cdot h(Y)] = \mathbb{E}[g(X + Y) \cdot h(Y)]$ for all continuous and measurable function h, which can be proved by Fubini Theorem.