Part II Dynamic optimization.

We consider a controlled dynamic system:

$$\boxed{X_{n+1} = f_n(X_n, u_n)}$$

Static optimization

$$\min_{x \in K} f(x) \qquad K \subseteq \mathbb{R}^n.$$

n: time. $n = 0, 1, 2, \dots$

X_n: state of the system at time n. $\quad X_n \in X \subseteq \mathbb{R}^d$

u_n: the decision / the control chosen at time n. $\quad u_n \in U_n$

x_0: the initial state. $(X_0 = x_0)$ $\quad f_n : X \times U_n \to X$.

dynamic optimization problem.

1. Finite Horizon

$$\underset{(u_n)_{n=0,1,\dots}}{\cancel{\min}\ \inf} \left(\sum_{n=0}^{N-1} L_n(X_n, u_n) + g(X_N) \right)$$

2. Infinite horizon.

$$\underset{(u_n)_{n=0,1,\dots}}{\cancel{\min}\ \inf} \left(\sum_{n=0}^{+\infty} \beta^n L_n(X_n, u_n) \right). \quad \text{for } \underbrace{\beta}_{\text{discount factor.}} \in (0,1)$$

- $L_n : X \times U_n \to \mathbb{R}$.
- $g : X \to \mathbb{R}$.

$$\boxed{X_{n+1}^{k,x,u} = f_n(X_n^{k,x,u}, u_n)}$$

1. Finite horizon problem.

$$X_k^{k,x,u} = x.$$

Value function: $$V(k,x) := \inf_{\substack{(u_n)_{n=k,k+1,\dots} \\ X_k^{k,x,u}=x.}} \left(\sum_{n=k}^{N-1} L_n(X_n^{k,x,u}, u_n) + g(X_N^{k,x,u}) \right)$$

Theorem: (Dynamic programming): For all $k=0,1,\dots,N-1$, $x \in X$.

one has. $- \; V(k,x) = \inf_{u_k \in U_k} \left(L_k(x,u_k) + V(k+1, X_{k+1}^{k,x,u}) \right).$

$$= \inf_{u_k \in U_k} \left(L_k(x,u_k) + V(k+1, f_k(x,u_k)) \right)$$

$$\boxed{- \; V(N,x) = g(x).} \to \text{terminal value.}$$

Proof: ① "$\geq$" Let $u=(u_n)_{n=k,k+1,\dots}$ be given arbitrarily.

Then: $$\sum_{n=k}^{N-1} L_n(X_n^{k,x,u}, u_n) + g(X_N^{k,x,u})$$

$$= L_k(X_k^{k,x,u}, u_k) + \left(\sum_{n=k+1}^{N-1} L_n(X_n^{k,x,u}, u_n) + g(X_N^{k,x,u}) \right)$$

$$= L_k(x, u_k) + \sum_{n=k+1}^{N-1} L_n\left(X_n^{k+1, X_{k+1}, u}, u_n\right) + g\left(X_N^{k+1, X_{k+1}, u}\right)$$

$$\geq L_k(x, u_k) + V(k+1, X_{k+1}^{k,x,u}).$$

$$X_n^{k,x,u} = X_n^{k+1, X_{k+1}^{k,x,u}, u} \rightarrow \text{flow property.}$$

$$\begin{cases} X_k = x. \\ X_{k+1} = f_k(X_k, u_k) \\ X_{k+2} = f_{k+1}(X_{k+1}, u_{k+1}) \end{cases} \qquad \begin{cases} X_{k+1} = X_{k+1}^{k,x,u} = f_k(X_k, u_k) \\ X_{k+2} = f_{k+1}(X_{k+1}, u_{k+1}) \end{cases}$$

Taking infimum on both sides over u, it follows that

$$V(k,x) \geq \inf_{u_k \in U_k} \left(L_k(x, u_k) + V(k+1, X_{k+1}^{k,x,u}) \right).$$

② "$\leq$". Denote $\overline{V}(k,x) := \inf_{u_k \in U_k} \left(L_k(x, u_k) + V(k+1, X_{k+1}^{k,x,u}) \right)$

For any $\varepsilon > 0$, there exists $u_k^\varepsilon \in U_k$ s.t.

$$\overline{V}(k,x) + \varepsilon \geq L_k(x, u_k^\varepsilon) + V(k+1, X_{k+1}^\varepsilon)$$

with $X_{k+1}^\varepsilon := X_{k+1}^{k,x,u_k^\varepsilon}.$

Next, by the definition of $V(k+1,\cdot)$, there exists $u_{k+1}^{\varepsilon}, \dots, u_{N-1}^{\varepsilon}$,

s.t. $$V(k+1, X_{k+1}^{\varepsilon}) + \varepsilon \geq \sum_{n=k+1}^{N-1} L_n(X_n^{\varepsilon}, u_n^{\varepsilon}) + g(X_N^{\varepsilon})$$

with $X_n^{\varepsilon} := X_n^{k+1, X_{k+1}^{\varepsilon}, u^{\varepsilon}}$.

$$\Longrightarrow \overline{V}(k,x) + \varepsilon + \cancel{V(k+1, X_{k+1}^{\varepsilon})} + \varepsilon$$

$$\geq L_k(x, u_k^{\varepsilon}) + \cancel{V(k+1, X_{k+1}^{\varepsilon})} + \sum_{n=k+1}^{N-1} L_n(X_n^{\varepsilon}, u_n^{\varepsilon}) + g(X_N^{\varepsilon})$$

$$\Longrightarrow \overline{V}(k,x) + 2\varepsilon \geq \sum_{n=k}^{N-1} L_n(X_n^{\varepsilon}, u_n^{\varepsilon}) + g(X_N^{\varepsilon}) \geq V(k,x).$$

By the arbitrariness of $\varepsilon > 0$, one has

$$V(k,x) \leq \overline{V}(k,x) = \inf_{u \in U_k} \left(L_k(x, u_k) + V(k+1, X_{k+1}^{k,x,u_k}) \right). \quad \#$$

Additional conditions: X, U_n, $n = 0, 1, \dots, N-1$ are all metric spaces.

U_n: $n = 1, \dots, N-1$ are compact.

$$\begin{cases} f_n : X \times U_n \to X, \\ L_n : X \times U_n \to \mathbb{R} \end{cases}$$ are continuous.

$$g: X \to \mathbb{R}$$

Proposition: Assume the above additional conditions. Then.

① $x \mapsto V(k,x)$ is continuous for each $k=0,1,\cdots,N$.
and there exists $u_k^*(x)$, s.t. $V(k,x) = L_k(x, u_k^*(x)) + V(k+1, X_{k+1}^{k,x,u_k^*(x)})$
for $k=0,1,\cdots,N-1$.

② Define $\bar{X}_0 = x$, and $\bar{X}_{n+1} = f_n(\bar{X}_n, \bar{u}_n)$ with $\bar{u}_n := u_n^*(\bar{X}_n)$.
Then, $(\bar{X}_n, \bar{u}_n)_{n=0,1,\cdots,N-1}$ is an optimal solution to the dynamic optimization problem. i.e.

$$V(0,x) = \sum_{n=0}^{N-1} L_n(\bar{X}_n, \bar{u}_n) + g(\bar{X}_N).$$

Proof: ①. – $V(N,x) := g(x)$ is a continuous function.

– Assume that $x \mapsto V(k+1,x)$ is continuous. then by Thm. of DPP.

$V(k,x) = \inf_{u \in U_k} \ell_k(x,u)$ where $\ell_k(x,u) := L_k(x,u) + V(k+1, f_k(x,u))$
is a continuous function in (x,u)

Recall that U_k is compact.

Then, for all $x \in X$, there exists $u_k^*(x) \in U_k$.

s.t. $V(k,x) = \ell_k(x, u_k^*(x))$

and we claim that $x \mapsto V(k,x)$ is also continuous

Indeed, let $x_n \to x$, then $V(k,x_n) \le \ell_k(x_n, u_k^*(x)) \xrightarrow{n\to+\infty} \ell_k(x, u_k^*(x))$

$= V(k,x)$.

and $\lim_{n\to+\infty} V(k,x_n) = \lim_{n\to+\infty} \ell_k(x_n, u_k^*(x_n)) = \ell_k(x, \bar{u}^*) \ge V(k,x)$.

$\Longrightarrow \lim_{n\to+\infty} V(k,x_n) = V(k,x)$, i.e. $x \mapsto V(k,x)$ is continuous.

② To prove that $(\bar{X}_n, \bar{u}_n)_{n=0,1,\cdots}$ is an optimal solution,

it is enough to prove that

$$(*) \quad V(0,x_0) = \sum_{n=0}^{m-1} L_n(\bar{X}_n, \bar{u}_n) + V(m, \bar{X}_m), \quad \forall\, m = 1, 2, \cdots, N.$$

Indeed, (*) is true for $m=0$

Next, assume that (*) is true for m. i.e. $V(0,x_0) = \sum_{n=0}^{m-1} L_n(\bar{X}_n, \bar{u}_n)$

$+ V(m, \bar{X}_m)$

Since $V(m, \bar{X}_m) = \ell(\bar{X}_m, \bar{u}_m)$

$$= L_m(\bar{X}_m, \bar{u}_m) + V(m+1, \bar{X}_{m+1})$$

$$\Rightarrow V(0, x_0) = \sum_{n=0}^{m-1} L_n(\bar{X}_n, \bar{u}_n) + L_m(\bar{X}_m, \bar{u}_m) + V(m+1, \bar{X}_{m+1})$$

$$= \sum_{n=0}^{m} L_n(\bar{X}_n, \bar{u}_n) + V(m+1, \bar{X}_{m+1})$$

$\Rightarrow$ (*) is also true for $m+1$.

By induction, (*) is true for all $m = 0, 1, \cdots, N$ #

2. Infinite horizon problem:

Assume that (f_n, L_n, U_n) are independent n.

$= (f, L, U)$ for all $n = 0, 1, \cdots$

L is uniformly bounded.

$X_{n+1} = f(X_n, u_n)$, $u_n \in U$.

$$\boxed{\inf_{(u_n)_{n=0,1,\cdots}} \sum_{n=0}^{+\infty} \beta^n L(X_n, u_n)} \qquad \beta \in (0,1)$$

Value function: $V(x) := \inf_{(u_n)_{n=0,1,\cdots}} \sum_{n=0}^{+\infty} \beta^n L(X_n^{0,x,u}, u_n)$

$0,x,u$

$$\begin{cases} X_0 = x. \\ \boxed{X_{n+1} = f(X_n, u_n)} \end{cases}$$

0	1	2
x	$X_1^{0,x,u} \to$	$X_2^{0,x,u}$.
	$\parallel$	
	$X_1^{1,X_1^{0,x,u}} \to$	$X_2^{1,X_1^{0,x,u}}$

Theorem (DPP) For all $x \in X$, one has

$$V(x) = \inf_{u_0 \in U} \left(L(x,u_0) + \beta V(X_1^{0,x,u_0}) \right)$$

$$= \inf_{u_0 \in U} \left(L(x,u_0) + \beta V(f(x,u_0)) \right).$$

Proof: ① Let $(u_n)_{n=0,1,\dots}$ be an arbitrary control process.

Then.
$$\sum_{n=0}^{+\infty} \beta^n L(X_n^{0,x,u}, u_n) = L(x,u_0) + \sum_{n=1}^{+\infty} \beta^n L(X_n^{0,x,u}, u_n)$$

$$X_n^{0,x,u}.$$

$$= L(x,u_0) + \beta \left(\sum_{n=1}^{+\infty} \beta^{n-1} L(X_n^{1,X_1^{0,x,u},u}, u_n) \right).$$

(flow property)

$$X_2^{1,X_1^{0,x,u},u} = f(X_1^{0,x,u}, u_1)$$

$$X_1^{0,X_1^{0,x,u},\bar{u}} = f(X_1^{0,x,u}, \bar{u}_0) = f(X_1^{0,x,u}, u_1)$$

$$\begin{array}{ccccccc}
X_1^{0,x,u} = X_1^{1,X_1^{0,x,u},u} & \xrightarrow{(f,u_1)} & X_2^{1,X_1^{0,x,u},u} & \xrightarrow{(f,u_2)} & X_3^{1,X_1^{0,x,u},u}. \\
\parallel \qquad \parallel & & \parallel & & \parallel \\
X_1^{0,x,u} = X_0^{0,X_1^{0,x,u},\bar{u}} & \xrightarrow{(f,\bar{u}_0)} & X_1^{0,X_1^{0,x,u},\bar{u}} & \xrightarrow{(f,\bar{u}_1)} & X_2^{0,X_1^{0,x,u},\bar{u}}.
\end{array}$$

$$\bar{u}_n = u_{n+1}$$

$$= L(x,u_0) + \beta\left(\sum_{n=1}^{+\infty} \beta^{n-1} L\left(X_{n-1}^{0,X_1^{0,x,u},\bar{u}}, \bar{u}_{n-1}\right)\right)$$

$$= L(x,u_0) + \beta\left(\underline{\sum_{n=0}^{+\infty} \beta^{n} L\left(X_{n}^{0,X_1^{0,x,u},\bar{u}}, \bar{u}_{n}\right)}\right)$$

$$\geq L(x,u_0) + \beta \cdot \underline{V\left(X_1^{0,x,u}\right)}$$

Taking infimum on both sides over u, it follows that

$$V(x) \geq \underbrace{\inf_{u_0 \in U}\left(L(x,u_0) + \beta V(\underbrace{f(x,u_0)}_{X_1^{0,x,u_0}})\right)}_{\triangleq \bar{V}(x)}$$

(2) For any $\varepsilon > 0$

there exists u_0^ε s.t. $\bar{V}(x) + \varepsilon \geq L(x, u_0^\varepsilon) + \beta V(X_1^{0,x,u_0^\varepsilon})$

Further, there exist $(u_1^\varepsilon, u_2^\varepsilon, \dots)$ s.t.

$$V(X_1^{0,x,u_0^\varepsilon}) + \varepsilon \geq \underline{\sum_{n=1}^{+\infty} \beta^{n-1} L\left(X_{n-1}^{0,X_1^{0,x,u_0^\varepsilon}, u^\varepsilon}, u_n^\varepsilon\right)}$$

$$\Longrightarrow \bar{V}(x) + 2\varepsilon \geq \sum_{n=0}^{+\infty} \beta L\left(X_n^{0,x,u^\varepsilon}, u_n^\varepsilon\right).$$

Since ε is arbitrary, it follows that

$$\underline{V}(x) \le \sum_{n=0}^{+\infty} \beta L(X_n^{0,x,u^\varepsilon}, u_n^\varepsilon) \le \overline{V}(x) = \inf_{u_0 \in U} \left(L(x, u_0) + \beta V(X_1^{0,x_0,u_0}) \right)$$

(DP equation): $$V(x) = \inf_{u_0 \in U} \left(L(x, u_0) + \beta V(f(x, u_0)) \right)$$

Theorem: Assume that L is uniformly bounded, $\beta \in (0,1)$.
Then, there exists a unique solution $V: X \to \mathbb{R}$ to the DP equation in the space of $B(X) := \{$ all functions $h: X \to \mathbb{R}$ bounded$\}$.

Remark: Let $\|h\|_\infty := \sup_{x \in X} |h(x)|$. then $(B(X), \|\cdot\|_\infty)$ is a Banach space.

Proof: ① Let $T: B(X) \to B(X)$ be defined by

$$T(h)(x) := \inf_{u_0 \in U} \left(L(x, u_0) + \beta h(f(x, u_0)) \right)$$

So that the DP equation is $V = T(V)$

$V = T(V)$,

i.e. a fixed point of T.

(2) It is enough to prove that $T: B(X) \to B(X)$ is a contraction mapping.

Let $h_1, h_2 \in B(X)$. then $T(h_1)(x) - T(h_2)(x)$

$$= \inf_{u_0 \in U} \left(L(x,u_0) + \beta h_1(f(x,u_0)) \right) - \inf_{u_0 \in U} \left(L(x,u_0) + \beta h_2(f(x,u_0)) \right).$$

$$\le \cancel{L(x, u_0^{2,\varepsilon})} + \beta h_1(f(x, u_0^{2,\varepsilon})) - \left(\cancel{L(x, u_0^{2,\varepsilon})} + \beta h_2(f(x, u_0^{2,\varepsilon})) \right) + \varepsilon.$$

$$\le \beta \|h_1 - h_2\|_\infty \cancel{+\varepsilon}.$$

Similarly, $T(h_2)(x) - T(h_1)(x) \le \beta \|h_1 - h_2\|_\infty$

$\Rightarrow \|T(h_1) - T(h_2)\|_\infty \le \beta \|h_1 - h_2\|_\infty$. for $\beta \in (0,1)$.

Therefore, T is a contraction mapping. #