Take any $x \in K$ $(x \neq x_0)$. and let $\boxed{v := x_0 - x \neq 0.}$

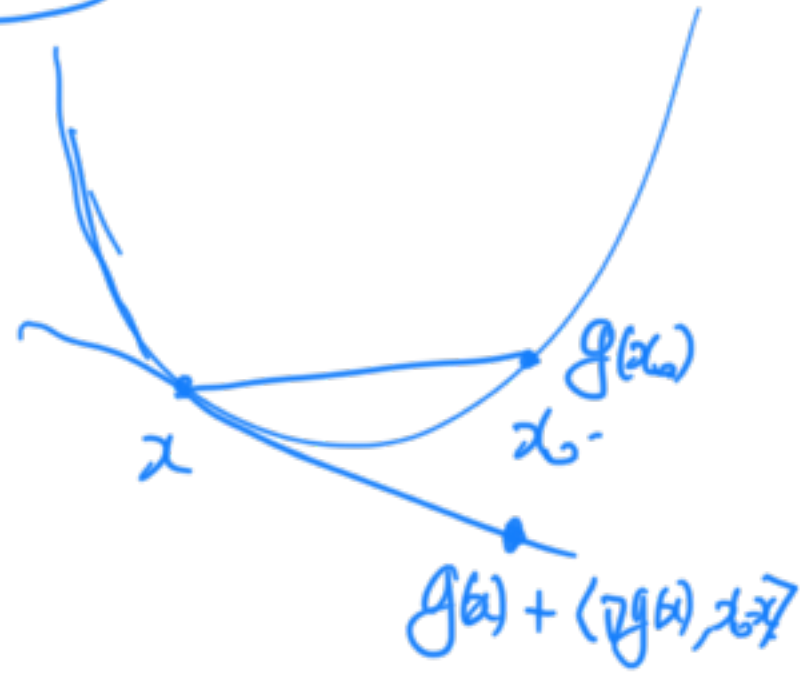
We will check that either $g_i(x) < 0$ or $\langle v, \nabla g_i(x) \rangle < 0$, ($g_i(x) = 0$.) $\forall\, i = 1, \cdots, m$

Indeed, as $g_i(x)$ is convex, if $g_i(x) = 0$.

then $g_i(x_0) - g_i(x) \geq \langle \nabla g_i(x), x_0 - x \rangle$ (convexity of g_i.)

$\Leftrightarrow 0 > g_i(x_0) \geq \langle \nabla g_i(x), v \rangle$.

$\Rightarrow \langle \nabla g_i(x), v \rangle < 0$.



Thus the Qualif. Cond. holds. #

Corrollary: Let x^* be a solution to (P_c). and ~~x^*~~ the constraints $g_i(x) \leq 0$. $i = 1, \cdots, m$. satisfy the qualification condition at x^*. Then, there exist $\lambda^* \in \mathbb{R}_+^m$ s.t.

$$
\begin{cases}
① \quad \sum_{i=1}^m \lambda_i^* g_i(x^*) = 0. \iff \lambda_i^* g_i(x^*) = 0, \quad \forall\, i = 1, \cdots, m. \\
② \quad \nabla_x L(x^*, \lambda^*) = 0. \quad \text{where } \boxed{L(x, \lambda) := f(x) + \sum_{i=1}^m \lambda_i g_i(x).}
\end{cases}
$$

3.2. Necessary and Sufficient condition.

Lemma: Let $F: \mathbb{R}^n \to \mathbb{R}$ be a convex function, of class C^1.

Then x^* be a solution to $\min_{x \in \mathbb{R}^n} F(x)$. iff $\nabla F(x^*) = 0$.

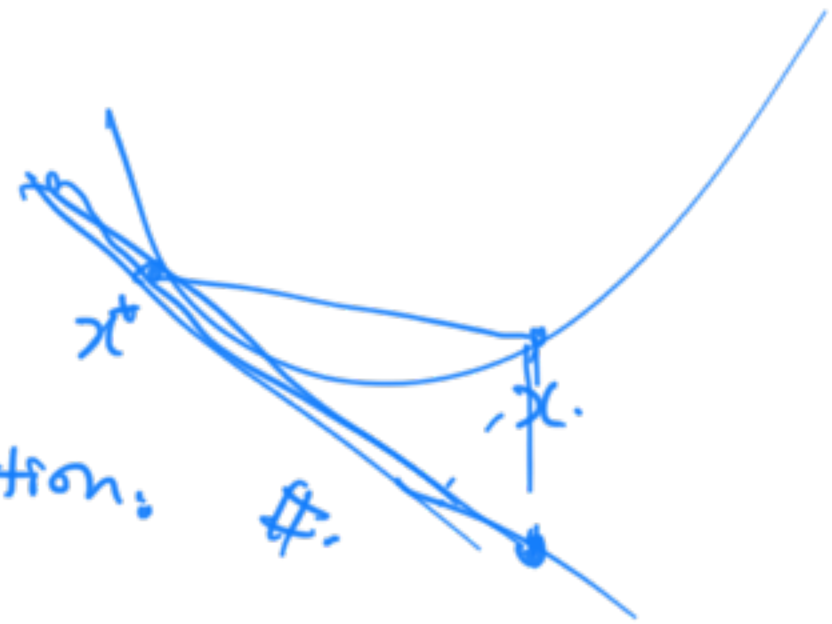
Proof: ① First, $\nabla F(x^*) = 0$ is a first order Necessary condition.

② Next, if x^* satisfies $\nabla F(x^*) = 0$.

then $F(x) \geq F(x^*) + \underline{\nabla F(x^*)} \cdot (x - x^*)$

$= F(x^*)$, $\forall x \in \mathbb{R}^n$.

Then. $\nabla F(x^*)$ is also a suffisant condition.



Theorem: Assume that the constraints in (P_c) satisfy the qualification condition at each point $x \in K$. Then, $x^* \in K$ is a solution to (P_c).

iff $\exists \underline{\lambda^* \in \mathbb{R}^m_+}$, s.t. $\begin{cases} \sum_{i=1}^m \lambda_i^* \cdot g_i(x^*) = 0 \\ \nabla_x L(x^*, \lambda^*) = 0 \end{cases}$ (*)

Proof: We only need to prove that (*) is also a suffisant condition to ensure that x^* is a solution to (P_c).

Assume that x^* satisfies (*), notice that $x \mapsto L(x, \lambda) = \underline{f(x) + \sum \lambda_i g_i(x)}$ is convex

Then. $\nabla_x L(x^*, \lambda^*) = 0$.

$\nabla_x L(x, \lambda^*) = 0$

implies that x^* is a solution to ~~the opt~~ $\min_{x \in \mathbb{R}^n} L(x, \lambda^*)$

$$\Rightarrow \underline{L(x^*, \lambda^*)} \le L(y, \lambda^*), \quad \forall y \in \cancel{\mathbb{R}^n}\ K.$$

$$L(x^*, \lambda^*) = f(x^*) + \sum_{i=1}^{m} \lambda_i^* g_i(x^*) = 0 \text{ by } (*)$$

$$= f(x^*)$$

$$L(y, \lambda^*) = f(y) + \underbrace{\sum_{i=1}^{m} \underbrace{\lambda_i^*}_{\ge 0} \underbrace{g_i(y)}_{\le 0}}_{\le 0} \le f(y), \quad \forall y \in K.$$

$$\Rightarrow \underline{f(x^*)} = L(x^*, \lambda^*) \le L(y, \lambda^*) \le \underline{f(y)}, \quad \forall y \in K.$$

$\Rightarrow$ x^* is a solution to (P_c). #.

3.3. Duality.

$$L(x, \lambda) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

$\underline{(D_c)}$. $\sup_{\lambda \in \mathbb{R}^m_+} d(\lambda)$, where $d(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda)$

Theorem. Assume that "$\underline{K}$" satisfy the qualification condition at each $x \in K$. and that $\underline{(P_c)}$ has a solution. $\underline{x^* \in K}$.

Then: $\min_{x \in K} f(x) = \inf_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^m_+} L(x, \lambda)$. (P_c)

$$= \sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} L(x,\lambda) \quad (D_c)$$

$$= \sup_{\lambda \in \mathbb{R}^m_+} d(\lambda) \Rightarrow \lambda^*.$$

Moreover, the dual problem (D_c) has a solution $\lambda^* \in \mathbb{R}^m_+$, and x^* is also a solution to $\inf_{x \in \mathbb{R}^n} L(x, \lambda^*) \Rightarrow d(\lambda^*)$.

Proof: ① $\sup_{\lambda \in \mathbb{R}^m_+} L(x,\lambda) = \sup_{\lambda \in \mathbb{R}^m_+} \left(f(x) + \sum \lambda_i g_i(x) \right) = \begin{cases} f(x) & \text{if } g_i(x) \le 0 \ \forall i = 1, \dots, m \iff x \in K. \\ +\infty & \text{if } x \notin K. \end{cases}$

$$\Rightarrow \min_{x \in K} f(x) = \min_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^m_+} L(x,\lambda)$$

$\hookrightarrow$ penalization on the constraints.

② weak duality: $\inf_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^m_+} L(x,\lambda) \ge \sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} L(x,\lambda)$:

Indeed, $\forall x \in \mathbb{R}^n$, $\left(\sup_{\lambda \in \mathbb{R}^m_+} L(x,\lambda) \right) \ge \left(L(x,\mu) \right)$, $\forall \mu \in \mathbb{R}^m_+$.

$$\Rightarrow \inf_{x \in \mathbb{R}^n} \left(\sup_{\lambda \in \mathbb{R}^m_+} L(x,\lambda) \right) \ge \inf \left(L(x,\mu) \right) \quad \forall \mu \in \mathbb{R}^m_+.$$

$$\Longrightarrow \underbrace{\inf_{x\in\mathbb{R}^n} \sup_{\lambda\in\mathbb{R}^m_+} L(x,\lambda)} \geq \sup_{\mu\in\mathbb{R}^m_+} \left(\underline{\inf_{x\in\mathbb{R}^n} L(x,\mu)} \right).$$

$$\Longrightarrow \inf_{x\in\mathbb{R}^n} \sup_{\lambda\in\mathbb{R}^m_+} L(x,\lambda) \geq \underline{\sup_{\lambda\in\mathbb{R}^m_+} \inf_{x\in\mathbb{R}^n} L(x,\lambda)}$$

③ Let x^* be a solution to (P_c).

Then, the constraints satisfy the qualification condition at x^*.

$$\Longrightarrow \exists\ \lambda^* \in \mathbb{R}^m_+ \ \text{s.t.} \begin{cases} \underline{\sum_{i=1}^m \lambda_i^* g_i(x^*) = 0} \\ \underline{\nabla_x L(x^*,\lambda^*) = 0.} \end{cases}$$

$$\Longrightarrow \underline{\sup_{\lambda\in\mathbb{R}^m_+} \left(\inf_{x\in\mathbb{R}^n} L(x,\lambda) \right)} \boxed{\geq} \inf_{x\in\mathbb{R}^n} L(x,\lambda^*)$$

by $\nabla_x L(x^*,\lambda^*)=0$

$$= L(x^*,\lambda^*) = f(x^*) + \underbrace{\sum \lambda_i^* g_i(x^*)}_{=0}$$

$$= f(x^*)$$

$$= \inf_{x\in K} f(x).$$

$$= \inf_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^m_+} L(x, \lambda).$$

In particular: ① one has: $\sup_{\lambda \in \mathbb{R}^m_+} \left(\inf_{x \in \mathbb{R}^n} L(x,\lambda) \right) = \inf_{x \in \mathbb{R}^n} L(x, \lambda^*)$

$$\Longleftrightarrow \underbrace{\sup_{\lambda \in \mathbb{R}^m_+} d(\lambda)}_{} = d(\lambda^*),$$

In other words, λ^* is a solution to the dual problem

$$\sup_{\lambda \in \mathbb{R}^m_+} d(\lambda) \quad (D_c)$$

② Moreover, x^* is a solution to $\inf_{x \in \mathbb{R}^n} L(x, \lambda^*)$

since $\nabla_x L(x^*, \lambda^*) = 0$. #.

Exercise:

$$\min_{\substack{x^2+y^2 \le 1 \\ y+z \le 0}} \underbrace{\frac{1}{2}\left[(x-2)^2 + y^2 + z^2 \right].}_{} \quad (P_c)$$

Solution: $f(x, y, z) = \frac{1}{2}\left[(x-2)^2 + y^2 + z^2 \right]$

$$\begin{cases} g_1(x,y,z) := x^2+y^2-1. \\ g_2(x,y,z) = y+z \end{cases}$$

① $f(x,y,z)$ is coersive. so that (P_c) has a solution (x^*, y^*, z^*)

and $K^< := \{(x,y,z) : g_1(x,y,z)<0,\ g_2(x,y,z)<0\} \neq \phi$. the Qualif. Cond. holds.

By duality: $(P_c) = \min_{(x,y,z)\in\mathbb{R}^3} \sup_{\lambda\in\mathbb{R}^2_+} f(x,y,z) + \lambda_1 g_1(x,y,z) + \lambda_2 g_2(x,y,z)$

$$= \sup_{\lambda\in\mathbb{R}^2_+} \underbrace{\inf_{(x,y,z)\in\mathbb{R}^3} f(x,y,z) + \lambda_1 g_1(x,y,z) + \lambda_2 g_2(x,y,z)}_{d(\lambda_1,\lambda_2).}$$

② $d(\lambda_1,\lambda_2) = \inf_{(x,y,z)\in\mathbb{R}^3} \cancel{L(x,y,z,\lambda_1,\lambda_2)}\ \frac{1}{2}\left[(x-2)^2+y^2+z^2\right]$

$$+ \lambda_1(x^2+y^2-1) + \lambda_2(y+z)$$

$$= \inf_{(x,y,z)} \Big((\tfrac{1}{2}+\lambda_1)x^2 - 2x + (\tfrac{1}{2}+\lambda_1)y^2 + \lambda_2 y$$

$$+\frac{1}{2}z^2+\lambda_2 z \qquad + \; 2-\lambda_1 \Big)$$

$$= -\frac{4}{4(1/2+\lambda_1)} - \frac{\lambda_2^2}{4(\frac{1}{2}+\lambda_1)} - \frac{1}{2}\lambda_2^2$$

$$+\; 2-\lambda_1$$

$$= \left(-\frac{4+\lambda_2^2}{2+4\lambda_1} - \frac{1}{2}(\lambda_2^2+4) - \lambda_1 \right)$$

$$\begin{cases} x^* = \dfrac{\cancel{2}}{\cancel{\frac{1}{2}+2\lambda_1}} = \dfrac{2}{1+2\lambda_1} \\ y^* = -\dfrac{\lambda_2}{1+2\lambda_1} \\ z^* = -\lambda_2 . \end{cases}$$

$$\max_{\substack{\lambda_1\geq 0\\ \lambda_2 \geq 0}} d(\lambda_1,\lambda_2) = \max_{\lambda_1 \geq 0}\left(-\frac{4}{2+4\lambda_1} - 2 - \lambda_1 \right), \quad \lambda_2^* = 0$$

$$= \max_{\lambda_1\geq 0}\left(-\frac{2}{1+2\lambda_1} - \boxed{\frac{1+2\lambda_1}{2}} \right) - \frac{3}{2} \qquad \boxed{\lambda_1^* = \frac{1}{2}.}$$

$$= -2-\frac{3}{2}$$

$$= \frac{7}{2}$$

$$\Rightarrow \boxed{\begin{cases} x^* = 1 \\ y^* = 0 \\ z^* = 0 \end{cases}}$$

$$(P_c): \quad \min_{x\in K} f(x) = \sup_{\lambda\in\mathbb{R}^m_+} \underbrace{\inf_{x\in\mathbb{R}^n} L(x,\lambda)}_{d(\lambda)}$$

Step 1: Compute $d(\lambda)$:

Step 2: Solve (D_c). Find λ^*.

Step 3: Find x^* as it is a solution to $d(\lambda^*)$.

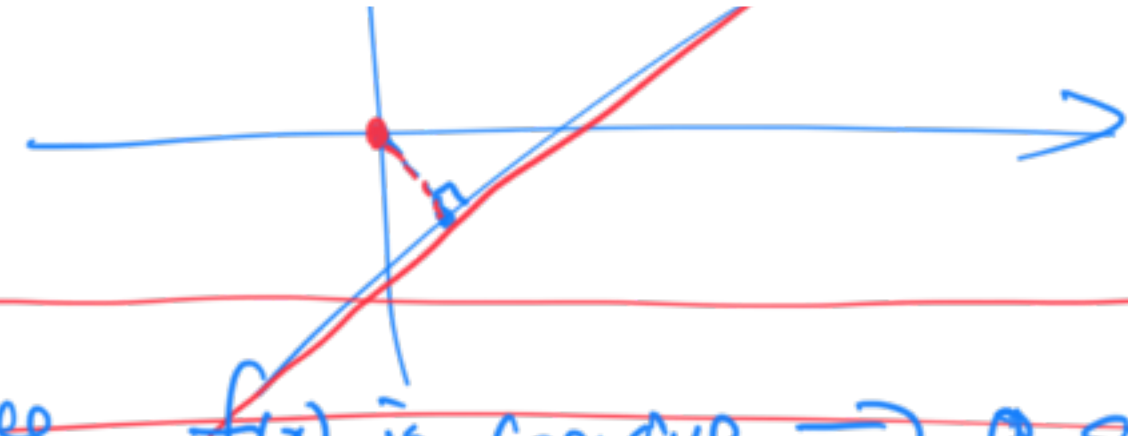
Exercise 2: $\min \frac{1}{2}\|x\|^2$
$\boxed{Ax=b}$

A is $m\times n$ matrice.
$b\in\mathbb{R}^m$.
$x\in\mathbb{R}^n$.

$\boxed{\text{Rank}(A)=m<n}$

① Remark: $m=1$, $n=2$. $\quad A_1x_1 + A_2x_2 = b$

Projection of 0 to the



space. $\{x\in\mathbb{R}^n : Ax=b\}$.

(2) Existence: $f(x)$ is coercive $\Longrightarrow$ a solution x^* exists.

(3) Duality: $L(x,\lambda) = f(x) + \langle \lambda, Ax-b\rangle$, $f(x) = \frac{1}{2}\|x\|^2$.

(3.1) $$\min_{Ax=b} f(x) = \inf_{x\in\mathbb{R}^n} \sup_{\lambda\in\mathbb{R}^m} L(x,\lambda).$$

↳ penalization on the constraints $Ax-b=0$

$$\geq \sup_{\lambda\in\mathbb{R}^m} \inf_{x\in\mathbb{R}^n} L(x,\lambda)$$

(3.2) Since x^* exists, and Qualif. Cond. holds

$\Longrightarrow \exists\, \lambda^*\in\mathbb{R}^m$ s.t. $\nabla_x L(x^*,\lambda^*) = 0$

$x\mapsto L(x,\lambda)$ is convex.

$$\Longrightarrow \sup_{\lambda\in\mathbb{R}^m} \inf_{x\in\mathbb{R}^n} L(x,\lambda) \geq \inf_{x\in\mathbb{R}^n} L(x,\lambda^*)$$

$$= L(x^*,\lambda^*) = f(x^*) + \underbrace{\lambda^*(Ax^*-b)}_{Ax^*=b}$$

$$= f(x^*)$$

$$= \inf_{x \in K} f(x)$$

$$= \inf_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^m} L(x, \lambda)$$

$$\Longrightarrow \sup_{\lambda \in \mathbb{R}^m} \inf_{x \in \mathbb{R}^n} L(x, \lambda) = \inf_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^m} L(x, \lambda)$$

In particular: λ^* is solution to $\sup_{\lambda \in \mathbb{R}^m} d(\lambda)$

with $d(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda)$

and x^* is solution to $\inf_{x \in \mathbb{R}^n} L(x, \lambda^*)$.

Solution to $\inf_{Ax=b} \frac{1}{2}\|x\|^2$.

$$\frac{1}{2}\|x - A^T\lambda\|^2 - \frac{1}{2}\|A^T\lambda\|^2$$

Step 1: Compute $d(\lambda) = \inf_{x \in \mathbb{R}^n} \left(\frac{1}{2}\|x\|^2 + \langle \lambda, Ax - b \rangle \right)$

$$= \inf_{x \in \mathbb{R}^n} \left(\frac{1}{2}\|x\|^2 + \langle A^T\lambda, x \rangle - \langle \lambda, b \rangle \right).$$

$$= -\frac{1}{2}\|A^\tau\lambda\|^2 - \langle\lambda, b\rangle. \qquad x = A^\tau\lambda.$$

Step 2: Solve the dual pb: $\sup_{\lambda\in\mathbb{R}^m} d(\lambda)$.

$$\Longleftrightarrow \sup_{\lambda\in\mathbb{R}^m}\left(-\frac{1}{2}\lambda^\tau A A^\tau\lambda - b^\tau\lambda\right).$$

$$\Longrightarrow \lambda^* = (AA^\tau)^{-1} b. \Longleftarrow AA^\tau\lambda^* + b = 0$$

$$d(\lambda^*) = -\frac{1}{2}\lambda^{*\tau}\cdots\cdots$$

$\mathrm{Rank}(A) = m$

$\Longrightarrow \mathrm{Rank}(AA^\tau) = m.$

Step 3: $x^* = A^\tau\lambda^* = A(AA^\tau)^{-1} b$

$L(x,\lambda) = f(x) + \lambda(Ax-b)$

Convex in x.

$$f(x^*) = \frac{1}{2}\|x^*\|^2 = \frac{1}{2}\|A(AA^\tau)^{-1} b\|^2.$$

#.

Ex 3: $\min_{x^\tau A x \le 1} \langle c, x\rangle$

A is symetric strictly positive $n\times n$ matrix.

$c\in\mathbb{R}^n$. $x\in\mathbb{R}^n$.

$K = \{x : x^\tau A x - 1 \le 0\}$ is compact.

$\Rightarrow$ x^* exists.

$$K^< := \{ x : \underline{x^T A x - 1} < 0 \} \neq \phi$$

$\Rightarrow$ The constraints are qualified.

$$d(\lambda) = \min_{x \in \mathbb{R}^n} \left(\langle c, x \rangle + \lambda (x^T A x - 1) \right) \qquad \lambda \geq 0.$$

$$= -\frac{1}{2\lambda} c^T A^{-1} c - \lambda. \qquad \underline{c + 2\lambda A x^* = 0}$$

$$x^* = \frac{-1}{2\lambda} A^{-1} c.$$

$$\sup_{\lambda \geq 0} d(\lambda) \Rightarrow \lambda^* = \sqrt{\frac{c^T A^{-1} c}{2}}$$

$$\Rightarrow x^* = -\frac{1}{\sqrt{2 c^T A^{-1} c}} A^{-1} c.$$

$$\Rightarrow f(x^*) = c^T x^* = -\frac{1}{\sqrt{2 c^T A^{-1} c}} c^T A^{-1} c$$

$$= \sqrt{2} \sqrt{\cdots}$$