$$\underbrace{\sup_{\alpha} \mathbb{E}\left[\int_0^T L(s, X_s^\alpha, \alpha_s)\,ds + g(X_T^\alpha)\right]}$$ $\longrightarrow$ subject to constraints.

Assesement: $\rightarrow$ Exam 50%

$\rightarrow$ Project (group of 2 persons.) $\rightarrow$ oral presentation. / short report. } 50%

I. Static Optimization Problem

$$(P): \quad \inf_{x \in K} f(x). \qquad \text{where } f: \mathbb{R}^n \to \mathbb{R}$$

$$K := \{x \in \mathbb{R}^n : g_i(x) \le 0, \ h_j(x) = 0\}$$
$$i = 1, \cdots, m. \qquad j = 1, \cdots, \ell.$$

$$\cancel{\left(\min_{x \in K} f(x)\right)}$$

Remark: If $m = \ell = 0$, then $K = \mathbb{R}^n$.

1. Existence of solution.

- Definition: Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function, and $K \subseteq \mathbb{R}^n$ be ~~of~~ nonempty.
  We say the infimum of f on K is the real value $l \in [-\infty, \infty)$
  s.t. ① $l \le f(x)$, $\forall x \in K$.
  ② $\exists (x_n)_{n\ge1} \subseteq K$ s.t. $\lim_{n\to+\infty} f(x_n) = l$.

- Remark: ① We denot $l = \inf_{x\in K} f(x)$, which is always well defined.
  ② we call $(x_n)_{n\ge1}$ a minimization sequence.

- Definition: If $\inf_{x\in K} f(x) > -\infty$, and there exists $x^* \in K$ s.t. $f(x^*) = \inf_{x\in K} f(x)$.
  Then we say $x^* \in K$ is a solution to (P).
  In this case, we write. $f(x^*) = \min_{x\in K} f(x)$.

Example: $K = (0,1)$, $f(x) = x$. then $\inf_{x\in K} f(x) = 0$. but x^* does not exists in K.

Remark: If $f: \mathbb{R}^n \to \mathbb{R}$ is continuous, and K is compact.

Then, a minimizer x^* exists..

Definition: We say a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is coersive.

Iff $\lim_{\|x\| \to +\infty} f(x) = +\infty$

Example: ① $f(x) = x^T A x + b^T x + c$. where A is definitely positive matrix $b \in \mathbb{R}^n$, $c \in \mathbb{R}$.

Then f is coersive.

② $f(x) = \frac{1}{x}$. $K = (0, +\infty)$. then f is not coersive.

Proposition: If $f_i: \mathbb{R} \to \mathbb{R}$ is bounded from below, and coersive for each $i = 1, \cdots, n$.

Then $f(x) := \sum_{i=1}^{n} f_i(x_i)$ $\quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$.

is coersive.

Proof: W.l.o.g., we assume that $f_i \geq 0$.

Assume that $(x_m)_{m\geq 1}$ is sequence s.t. $\underbrace{\|x_m\|}_{\sqrt{\sum_{i=1}^{n} x_{m,i}^2}} \to +\infty$, as $m \to +\infty$

$x_m = (x_{m,1}, \cdots, x_{m,n}) \in \mathbb{R}^n$.

Then for some $i \in \{1, \cdots, n\}$ one has $|x_{m,i}| \to +\infty$ as $m \to +\infty$

Then: $f(x_m) \geq f_i(x_{m,i}) \longrightarrow +\infty$ as $m \to +\infty$. #

Theorem: Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous and coercive,

$K \subseteq \mathbb{R}^n$ be a closed nonempty subset.

Then (P) has a solution.

Proof: let $M := \inf_{x \in K} f(x) < +\infty$, then there exists $L > 0$

s.t. $f(x) \geq M$ for all $x \in K$ & $\underline{\|x\| \geq L}$.

Therefore, $\inf_{x \in K} f(x) = \inf_{x \in K_L} f(x)$. with $K_L := \{x \in K : \underline{\|x\| \leq L}\}$

Since K_L is bounded and closed, then K_L is compact.

Hence, there exists $x^* \in K_L$ s.t. $f(x^*) = \inf_{x \in K_L} f(x) = \inf_{x \in K} f(x)$. #

Proposition: Let K be an open and bounded subset in $\mathbb{R}^n$.

and f is continuous on $\overline{K} := \{x \in \mathbb{R}^n : \exists (x_n) \subseteq K \text{ s.t. } x_n \to x.\}$

↳ closure of K.

and there exists $x_0 \in K$ s.t. $f(x_0) \leq f(x)$ for all $x \in \partial K := \overline{K} \setminus K$.

Then: (P) has a solution $x^* \in K$.

Proof: First, $\overline{K}$ is a compact subset in $\mathbb{R}^n$. then $\inf_{x \in \overline{K}} f(x)$ has a solution $x^* \in \overline{K}$.

Next, If $x^* \in K$, then $f(x^*) = \inf_{x \in \overline{K}} f(x) = \inf_{x \in K} f(x)$.

If $x^* \in \partial K$, then $f(x_0) \leq f(x^*) = \inf_{x \in \overline{K}} f(x)$.

$\inf_{x \in K} f(x) \leq$

$\Longrightarrow \inf_{x \in K} f(x) = f(x_0) = f(x^*) = \inf_{x \in K} f(x)$ since $\inf_{x \in K} f(x) \geq \inf_{x \in \mathbb{R}} f(x)$

$\Longrightarrow x_0 \in K$ is solution to (P). #

2. Necessary condition of the optional solution x^*.

2.1. Euler condition.

Thm: Let $K \subseteq \mathbb{R}^n$ be an open set. $f: \mathbb{R}^n \to \mathbb{R}$ is in C^1.
Assume that (P) has a solution $x^* \in K$.
Then, $\nabla f(x^*) = 0$.

Proof: Let $e \in \mathbb{R}^n$, $\varepsilon > 0$ small enough so that $x_\varepsilon := x^* + \varepsilon e \in K$.
(since K is an open set).

$\Longrightarrow f(x_\varepsilon) \geq f(x^*)$. $\forall \varepsilon > 0$.

$$\Longrightarrow 0 \leq \lim_{\varepsilon \downarrow 0} \frac{f(x_\varepsilon) - f(x^*)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{f(x^* + \varepsilon e) - f(x^*)}{\varepsilon} = \langle \nabla f(x^*), e \rangle$$

$\Rightarrow \langle \nabla f(x^*), e \rangle \geq 0$, for all $e \in \mathbb{R}^n$.

$\Rightarrow \nabla f(x^*) = 0$. #.

2.2. Kuhn-Tucker Theorem.

- $K := \{ x \in \mathbb{R}^n : g_i(x) \leq 0 , i=1,\cdots,l. \quad h_j(x) = 0 , j=1,\cdots,m. \}$ $\quad I = \{1,\cdots,l\}$ $\quad J = \{1,\cdots,m\}$.

We will assume that $g_i, h_j \in C^1$. so that K is closed.
$f \in C^1$.

$(\mathcal{P})$: $\inf_{x \in K} f(x) \iff \inf_{x \in \mathbb{R}^n} f(x)$ subject to $\boxed{g_i(x) \leq 0}$, $i \in I$
$h_j(x) = 0$, $j = J$.

Theorem: Let x^* be a solution to $(\mathcal{P})$.
Then, there exists $p_0 \geq 0$, $\boxed{p \in \mathbb{R}^l_+}$, $q \in \mathbb{R}^m$. s.t.
① $(p_0, p, q) \neq 0$. ② $\sum_{i \in I} p_i g_i(x^*) = 0$.

③ $p_0 \nabla f(x^*) + \sum_{i\in I} p_i \nabla g_i(x^*) + \sum_{j\in J} q_j \nabla h_j(x^*) = 0$

Remark: ① If $p_0 > 0$. then ③ is equivalent to

$$\nabla f(x^*) + \sum_{i\in I} p_i \nabla g_i(x^*) + \sum_{j\in J} q_j \nabla h_j(x^*) = 0$$

Example: (p_0 could be 0).

$$\min_{x^2=0} x \qquad f(x) = x, \quad h(x) = x^2.$$

$\nabla f(x^*) = 1$, $\nabla h(x^*) = 2x^* = 0$, $\Rightarrow K := \{x : x^2 = 0\} = \{0\}$

$x^* = 0$.

$\Longrightarrow$ to ensure that $\boxed{p_0 \nabla f(x^*) + q \nabla h(x^*) = 0}$.

we need to set $p_0 = 0$.

② $\underbrace{\sum_{i\in I} p_i g_i(x^*)}_{\leq 0} = 0$, $p_i \geq 0$, $g_i(x^*) \leq 0$, $\forall i \in I$.

$\Longrightarrow p_i g_i(x^*) = 0$, $\forall i \in I$.

③ Formally, it is a first order necessary condition for x^*,

because: $\inf_{\substack{g(x)\le 0\\ h(x)=0}} f(x) \approx \inf_{\substack{g(x)\le 0\\ h(x)=0}} p_0 f(x)$

$$\Longleftrightarrow \inf_{\boxed{x\in\mathbb{R}^n}} \sup_{\substack{p\ge 0\\ q\in\mathbb{R}}} \left(p_0 f(x) + \underline{p\, g(x)} + \underline{q\, h(x)}\right)$$

penalization.

$$\| \begin{cases} p_0 f(x). & \text{if } g(x)\le 0 \text{ and } \underline{h(x)=0} \\ +\infty & \text{if } \underline{g(x)>0} \text{ or } h(x)\ne 0. \end{cases}$$

$$\langle\approx\rangle \sup_{\substack{p\ge 0\\ q\in\mathbb{R}}} \inf_{x\in\mathbb{R}^n} \left(p_0 f(x) + p\, g(x) + q\, h(x)\right)$$

Euler condition.

$$\longrightarrow p_0 \nabla f(x^*) + p \nabla g(x^*) + q \nabla h(x^*) = 0.$$

Proof. ① Let $f_N(x) := f(x) + \|x-x^*\|^2 + \frac{N}{2}\left(\sum_{i\in I} \max(0, g_i(x))^2 + \sum_{j\in J} h_j(x)^2\right)$.

it is clear that $f_N(x) \geq f(x) \quad \forall\ x \in \mathbb{R}^n$.

and $f_N(x^*) = f(x^*)$ since $x^* \in K \Leftrightarrow g_i(x^*) \leq 0,\ h_j(x^*) = 0$

and $f_N \in C^1$. Since $x \mapsto x_+^2$ is C^1.

and. $\min_{x\in K} f_N(x)$ has a unique solution x^*.

② Claim: $\exists\ \varepsilon_0 > 0$, s.t. $\forall\ \varepsilon \in (0, \varepsilon_0]$, $\exists\ N_\varepsilon > 0$.

s.t. $f_{N_\varepsilon}(x) > \underset{\underset{f(x^*)}{\|}}{f_{N_\varepsilon}(x^*)}$, $\forall\ \|x-x^*\| = \varepsilon$, $x \in \mathbb{R}^n$.

We consider the pb: $\min_{\|x-x^*\|<\varepsilon} f_{N_\varepsilon}(x)$, whose solution is not x^* a priori

which has a solution since $\{x : \|x-x^*\| < \varepsilon\}$ is open.

and $f_{N_\varepsilon}(x^*) < f_{N_\varepsilon}(x)$, $\forall\ x \in \partial K$. (claim)

Let x_ε^* be a solution to $\min_{\|x-x^*\|<\varepsilon} f_{N_\varepsilon}(x)$.

By Euler condition, $\nabla f_{N_\varepsilon}(x_\varepsilon^*) = 0$

$$\Longleftrightarrow 1 \cdot \nabla f(x_\varepsilon^*) + 2(x_\varepsilon^* - x^*) + N_\varepsilon \left(\sum_{i\in I} \boxed{\max(0, g_i(x_\varepsilon^*))} \cdot \nabla g_i(x_\varepsilon^*) + \sum_{j\in J} \boxed{h_j(x_\varepsilon^*)} \cdot \nabla h_j(x_\varepsilon^*) \right) = 0$$

$$\Longleftrightarrow \boxed{\varphi_0^\varepsilon \nabla f(x_\varepsilon^*) + 2\varphi_0^\varepsilon (x_\varepsilon^* - x^*) + \sum \varphi_i^\varepsilon \nabla g_i(x_\varepsilon^*) + \sum q_j^\varepsilon \nabla h_j(x_\varepsilon^*) = 0}$$

where, $\rho^\varepsilon := \sqrt{1 + N_\varepsilon^2 \sum \max(0, g_i(x_\varepsilon^*))^2 + N_\varepsilon^2 \sum h_j(x_\varepsilon^*)^2}$

$$\varphi_0^\varepsilon := \frac{1}{\rho^\varepsilon} > 0, \quad \varphi_i^\varepsilon := \frac{N_\varepsilon \cdot \max(0, g_i(x_\varepsilon^*))}{\rho^\varepsilon} \geq 0, \quad q_j^\varepsilon := \frac{N_\varepsilon h_j(x_\varepsilon^*)}{\rho^\varepsilon}$$

So that $\boxed{(\varphi_0^\varepsilon)^2 + \sum_{i\in I} (\varphi_i^\varepsilon)^2 + \sum_{j\in J} (q_j^\varepsilon)^2 = 1}$

Let $\varepsilon \downarrow 0$. then: $\|x_\varepsilon^* - x^*\| \leq \varepsilon \to 0. \iff x_\varepsilon^* \to x^*$.

along a subsequence, $\underline{(\varphi_0^\varepsilon, \varphi^\varepsilon, q^\varepsilon)} \longrightarrow \underline{(\varphi_0, \varphi, q)} \in \mathbb{R}_+ \times \mathbb{R}_+^\ell \times \mathbb{R}^m$

$$\Longrightarrow \boxed{\varphi_0 \nabla f(x^*) + \sum_{i\in I} \varphi_i \nabla g(x^*) + \sum_{j\in J} q_j \nabla h_j(x^*) = 0 \quad (3)}$$

and $\|(\varphi_0, \varphi, q)\| = 1. \Longrightarrow \boxed{(\varphi_0, \varphi, q) \neq 0. \quad (1)}$

Besides: If $g_i(x^*) < 0$, $\Longrightarrow \varphi_i^\varepsilon := N_\varepsilon \underline{\max(0, g_i(x_\varepsilon^*))} / \rho_\varepsilon = 0$

for $\varepsilon > 0$ small enough.

$$\Longrightarrow \varphi_i = 0.$$

Then: $\varphi_i g_i(x^*) = 0$, $\forall i \in I$. $\Longrightarrow \boxed{\sum_{i\in I} \varphi_i g_i(x^*) = 0. \quad (2)}$

(3) Let us finally prove the claim at the beginning of (2).

Assume that it is not true, then for $\varepsilon > 0$, for all $N > 0$.

$\exists\ x_N$ s.t. $\|x_N - x^*\| = \varepsilon$ and $f_N(x_N) \leq f(x^*)$

$\Rightarrow \{x_N\}_{N \geq 1}$ is a sequence in $\{x : \|x - x^*\| = \varepsilon\}$.

$\Rightarrow$ Along a subsequence, $x_N \to \bar{x}$

Besides, $\overline{\lim_{N \to +\infty}} \frac{N}{2} \left(\sum_{i \in I} \max(0, g_i(x_N))^2 + \sum_{j \in J} (h_j(x_N))^2 \right)$

$\overline{\lim_{N \to +\infty}} f_N(x_N) \leq f(x^*) < +\infty$

$$\Rightarrow \lim_{N \to +\infty} \left(\sum_{i \in I} \max(0, g_i(x_N))^2 + \sum_{j \in J} (h_j(x_N))^2 \right) = 0$$

$\Rightarrow \sum \max(0, g_i(\bar{x}))^2 + \sum (h_j(\bar{x}))^2 = 0. \Rightarrow g_i(\bar{x}) \leq 0,\ h_j(\bar{x}) = 0$

$\Rightarrow \bar{x} \in K. \quad \Rightarrow \underline{f(\bar{x})} + \underline{\|\bar{x}-x^*\|^2} \leq \varlimsup_{N\to+\infty} f_N(x_N) \leq \underline{f(x^*)}$

which is impossible since, $\|\bar{x}-x^*\| = \varepsilon$.

and $f(x^*) \leq f(\bar{x})$ as $\bar{x} \in K$. #