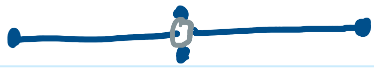
Lect16-20180312

Sunday, 11 March 2018 8:32 AM

Recollection $X = [-1,1] \sqcup [-1,1] / \sim$

$=$ 

$A =$  $\longleftrightarrow [-1,1]$

Both X, A are compact

$A \subset X$ is not closed

Fact. Given a compact X and $A \subset X$

A is closed $\Longrightarrow$ A is compact

$\cancel{\Longleftarrow}$??? condition

Which good property that  lacks?

 is not Hausdorff.

Theorem. Let X be Hausdorff and $A \subset X$.

A is closed $\Longleftarrow$ A is compact.

Proof will be given later.

Corollary. Let X be compact Hausdorff, $A \subset X$.

A is closed $\Longleftrightarrow$ A is compact

Lect16-p2

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Theorem. Let X be compact and Y be Hausdorff. If $f : X \longrightarrow Y$ is a continuous $\underbrace{\text{bijection}}_{f^{-1} \text{ exists}}$ then f is a $\underbrace{\text{homeomorphism.}}_{\text{continuity of } f^{-1} \text{ is automatic.}}$

Proof.

$$X \xleftarrow{\;f^{-1}\;} Y$$

$\cap$

Take any $U \in \mathcal{T}_X$ wish. $(f^{-1})^{-1}(U) \in \mathcal{T}_Y$

$$(f^{-1})^{-1}(U) = f(U)$$

$$\Updownarrow$$

Take any closed $F \subset X$ wish $f(F) \subset Y$ is closed

$\Downarrow$ X is compact

$\Uparrow$ Above: Y is Hausdorff

$$F \text{ is compact} \xRightarrow[\text{continuous image}]{} f(F) \text{ is compact}$$

Theorem. A is compact $\xrightarrow[A \subset X]{X \text{ is } T_2}$ A is closed.

Proof. To show A is closed

What can we do ?

Either $\boxed{\bar{A} \subset A}$ or $\boxed{X \setminus A \in \mathcal{T}_X}$

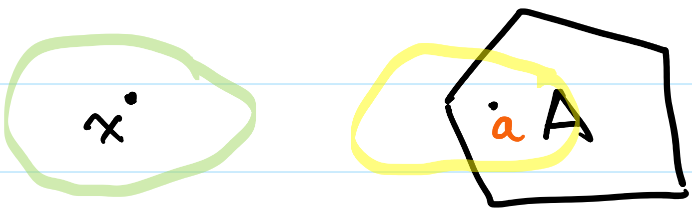
No idea

$x \in X \setminus A$, $a \in A$

use Hausdorff

Let $x \in X \setminus A$, for any $a \in A$

$\exists\ \underset{x}{\underset{\cup}{U_a}}, \underset{a}{\underset{\cup}{V_a}} \in \mathcal{T}_X$ such that $U_a \cap V_a = \phi$



Then $\mathcal{G} = \{V_a \subset X : a \in A\}$, $\cup\mathcal{G} \supset A$

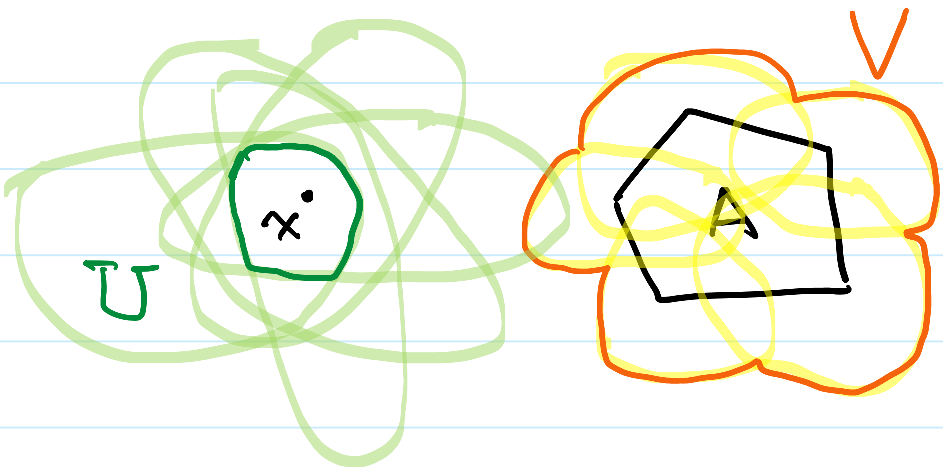
By compactness of A, $\exists\, a_1, a_2, \cdots, a_n \in A$

$$V = \bigcup\{V_{a_1}, V_{a_2}, \cdots, V_{a_n}\} \supset A$$

Correspondingly, we have

$$U = \bigcap\{U_{a_1}, U_{a_2}, \cdots, U_{a_n}\} \in \mathcal{T}_X$$

finite intersection



Obviously $x \in U \subset X \setminus A$
$U \in \mathcal{T}_X$

As $x \in X \setminus A$ is arbitrary, $X \setminus A \in \mathcal{T}_X$

Actually proved: Let X be Hausdorff.

$\forall$ compact $A \subset X$, $\forall\, x \notin A$

$\exists\ U, V \in \mathcal{T}_X$ such that

$x \in U$, $A \subset V$, $U \cap V = \phi$.

Lect16-p5

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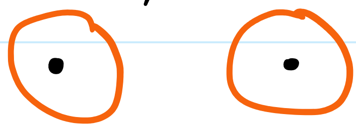
Separation Properties about a space $(X, \mathcal{T})$

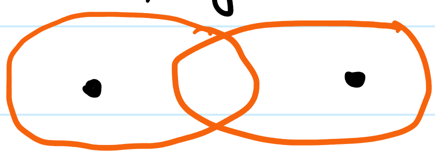
T_2 = Hausdorff: $\forall\, x \neq y\ \exists\, U, V \in \mathcal{T}$, $x \in U$, $y \in V$, $U \cap V = \emptyset$



$T_3 = T_1 +$ regular $\Rightarrow T_2$

$T_2 \Downarrow$

T_1 : $\forall\, x \neq y\ \exists\, U, V \in \mathcal{T}$
$x \in U \setminus V$, $y \in V \setminus U$



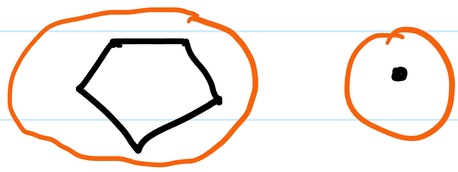
$\Updownarrow$ Every singleton is closed

$T_4 = T_1 +$ normal $\Rightarrow T_3$

$T_1 \Downarrow$

T_0 : $\forall\, x \neq y$, $\exists\, U \in \mathcal{T}$,
$x \in U$ & $y \notin U$ or
$x \notin U$ & $y \in U$

Regular: $\forall$ closed $F \subset X$, $\forall\, x \notin F$, $\exists\, U, V \in \mathcal{T}$
$F \subset U$, $x \in V$, $U \cap V = \emptyset$



Lect16-p6

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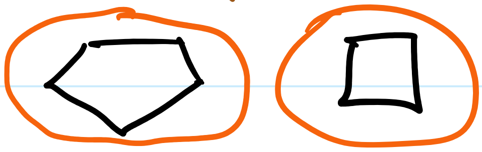
Normal: Write the definition as exercise

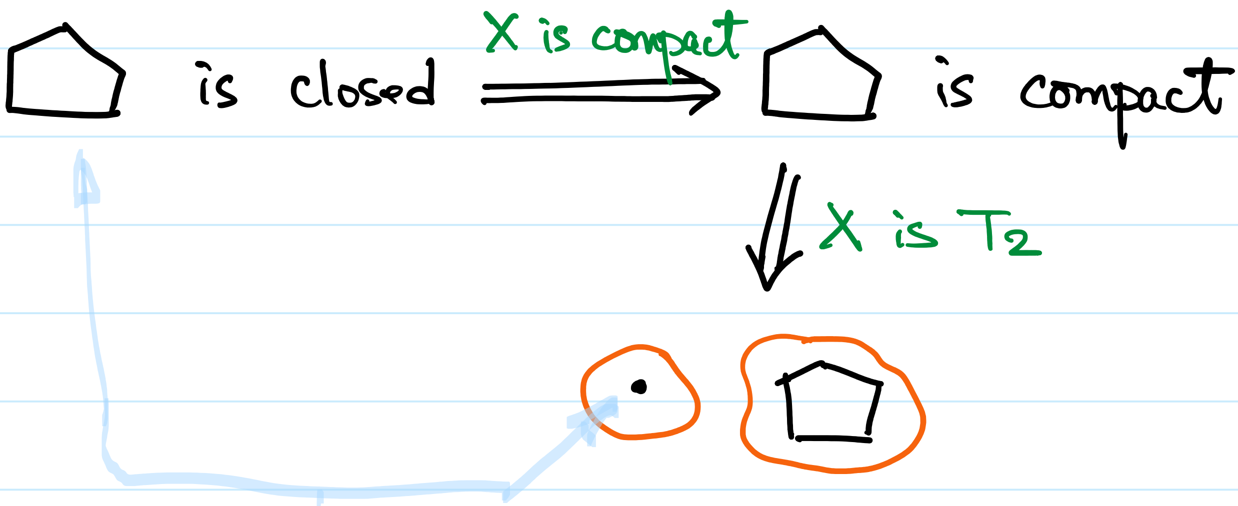


Most commonly used: T_1, T_2, T_4

Others: completely regular $= T_{3.5}$

perfectly regular $= T_6$

About compactness and separation, what is the overall picture?



arbitrary, thus X is regular + T_2

$\Downarrow$

T_3

Not yet finished, how to proceed?

Lect16-p7

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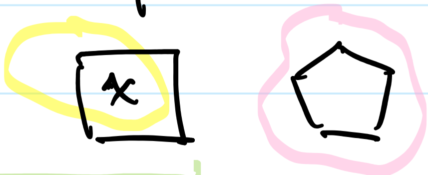
Strength of Compact Hausdorff space.

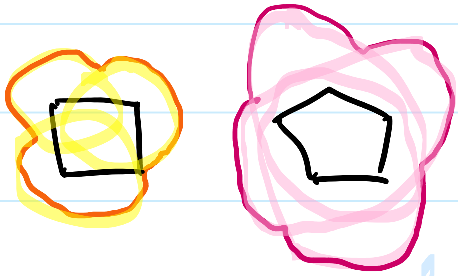
From above: automatically regular + T_2, cpt

Given a closed set ⬠ and another disjoint closed set □

$\xrightarrow{T_2}$ compact

$\forall x \in □$, 

determines an open cover for □, and finite subcover

Hence, for arbitrary disjoint closed □ & ⬠ $\Longrightarrow$ 

normal

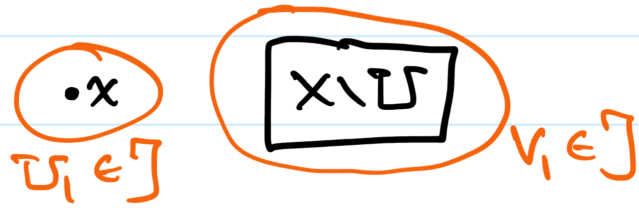
Conclusion. In a compact space, $T_2 \Leftrightarrow T_3 \Leftrightarrow T_4$

Advantage of Regular or Normal spaces

Take any $x \in U$ with $U \in \mathcal{J}$

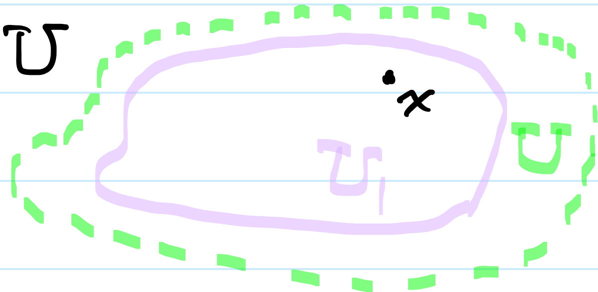
Then $x \notin X \setminus U$ — closed in X

X is regular, $\therefore \exists$



$U_1 \cap V_1 = \emptyset$ i.e., $U_1 \subset \underbrace{X \setminus V_1}_{\text{closed}} \subset U$

$\therefore \quad x \in U_1 \subset \overline{U_1} \subset U$



Iteratively,

$x \in \cdots U_n \subset \overline{U_n} \subset \cdots \subset U$

When X is normal, same argument works for any closed set $F \subset U \in \exists$ to have

$F \subset \cdots \subset U_n \subset \overline{U_n} \subset \cdots \subset U$

Tietz Extension Theorem

Exercise: write the statement

Urysohn Lemma. A space X is normal if and only if

$\forall$ closed $A, B \subset X$ with $A \cap B = \emptyset$

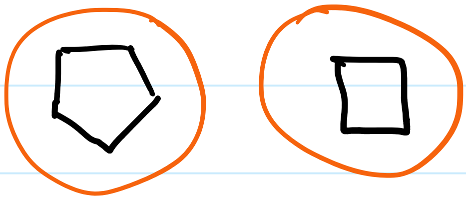
$\exists$ continuous $f : X \longrightarrow [-1, 1]$ such that

$f|_A \equiv -1$ and $f|_B \equiv 1$.

"$\Leftarrow$" Obvious.

Let $A, B \subset X$ be closed, $A \cap B = \phi$

wish:



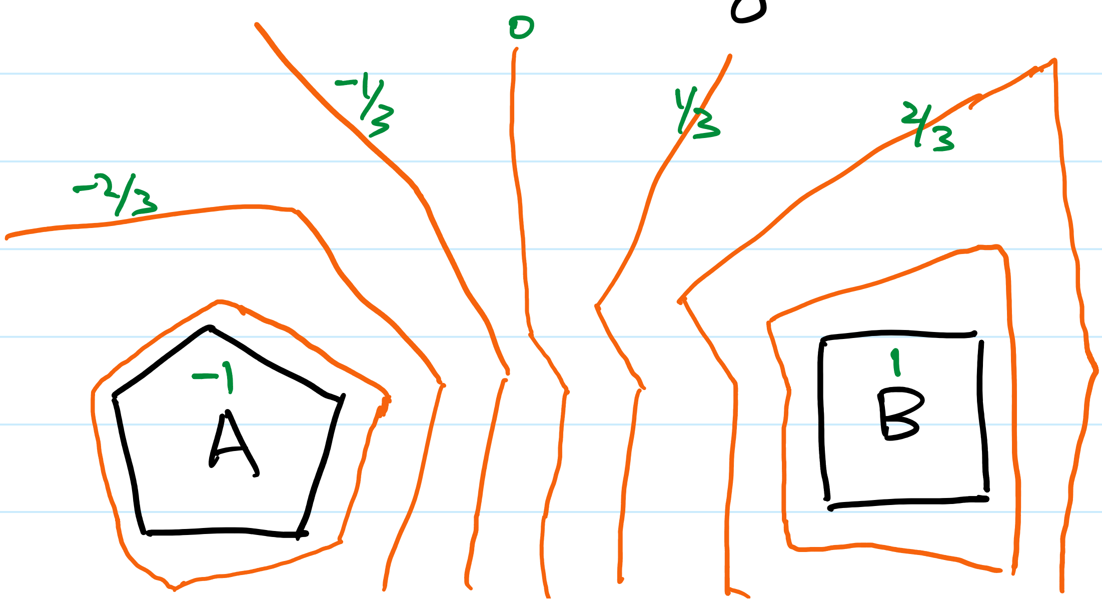
Just take $[-1, -\frac{1}{2})$ and $(\frac{1}{2}, 1]$, which are open in $[-1, 1]$. Then

$$\left.\begin{aligned} U &= f^{-1}[-1, -\tfrac{1}{2}) \supset A \\ V &= f^{-1}(\tfrac{1}{2}, 1] \supset B \end{aligned}\right\} \quad U \cap V = \phi$$

"Idea of $\Rightarrow$" Remember that we proved this for a metric space before, where

$$f(x) = \frac{d(x, A) - d(x, B)}{d(x, A) + d(x, B)}$$

The "level sets" of $f(x)$ may be helpful to our understanding



Start with $A \subset X\setminus B \overset{\text{def}}{=} U_1 \in \mathcal{T}$,

From previous discussion,

we have $A \subset U_{-1} \subset \underbrace{\underset{\text{closed}}{\overline{U_{-1}}} \subset \underset{\text{open}}{U_1}}_{\text{得寸進尺}} = X\setminus B$

How to proceed?

But then, where to work on next?

$$A \subset U_{-1} \subset \overbrace{\overline{U_{-1}} \subset V}^{} \overset{\text{or}}{\subset} \overbrace{\overline{V} \subset U_1}^{} = X\setminus B$$

We need a systematic procedure.

Write $\mathbb{Q} \cap (-1,1) = \{q_1, q_2, \dots, q_n, \dots\}$

steps

For $n=1$, take q_1, so $-1 < q_1 < 1$, get

$$A \subset \overline{U_{-1}} \subset U_{q_1} \subset \overline{U_{q_1}} \subset U_1 = X\setminus B$$

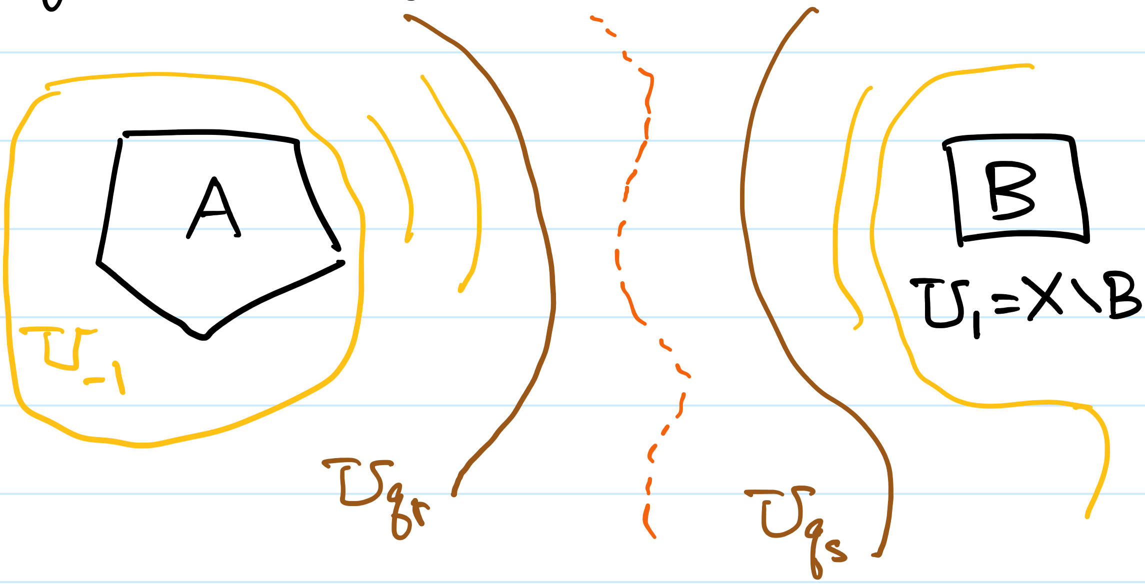
This started the following situation:

$\exists\ U_{q_1}, U_{q_2}, \dots, U_{q_n} \in \mathcal{T}$ such that

if $q_i < q_j$ then $\overline{U_{q_i}} \subset U_{q_j}$

Next, take $q_{n+1} \in \mathbb{Q} \cap (-1,1)$, let

$$q_r = \max\{q_1, \cdots, q_n < q_{n+1}\}$$
$$q_s = \min\{q_1, \cdots, q_n > q_{n+1}\}$$



By X is normal, get $U_{q_{n+1}}$ where

$$\begin{array}{l} U_{-1} \subset \cdots \subset \overline{U}_{q_r} \subset U_{q_{n+1}} \subset \overline{U}_{q_{n+1}} \subset U_{q_s} \subset \cdots \subset U_1 \\ \cup \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \parallel \\ A \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; X\setminus B \end{array}$$

$U_{q_1}, U_{q_2}, \cdots\cdots, U_{q_n}$

By this, we have the following.

- $\forall\, q \in \mathbb{Q} \cap [-1,1]$, $U_q \in \mathcal{T}$
- If $p, q \in \mathbb{Q} \cap [-1,1]$ and $p < q$

$$A \subset U_p \subset \overline{U}_p \subset U_q \subset U_1 = X\setminus B$$

Lect16-p12

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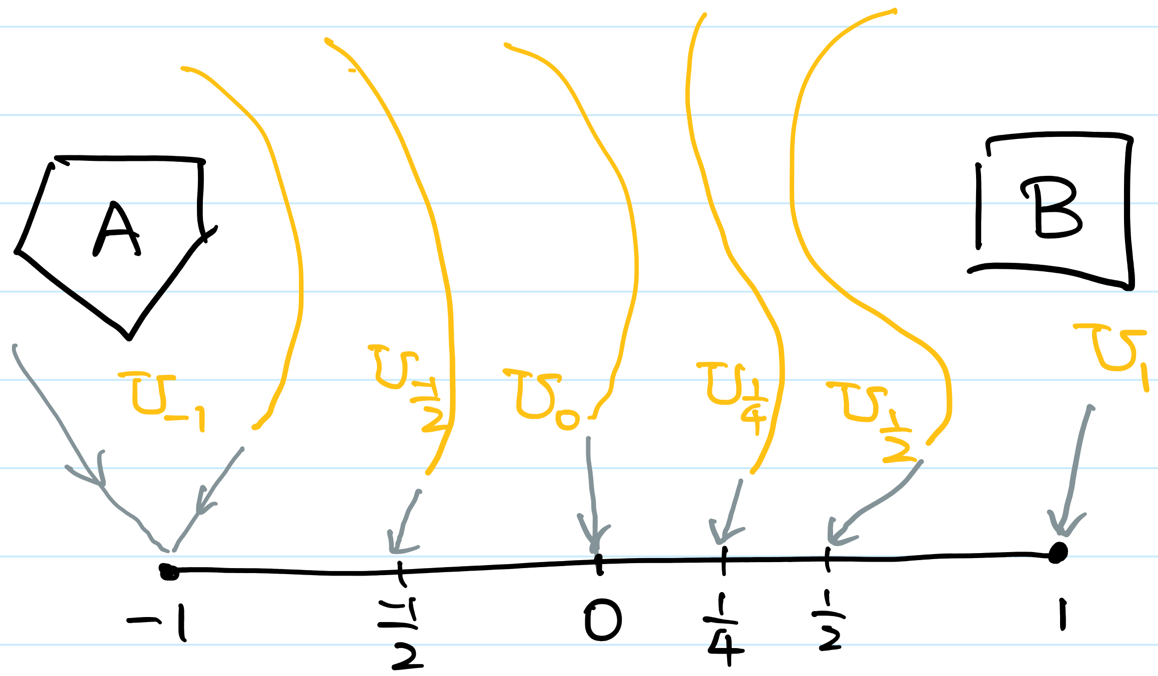
Define $f(x) = \inf\{q \in \mathbb{Q} \cap [-1,1] : x \in U_q\}$



Technical step:

Verify that f is continuous

Let $(a,b) \subset [-1,1]$ or $[-1,b)$ or $(a,1]$

Wish: $f^{-1}(a,b)$ is open

Take any $x \in f^{-1}(a,b)$, i.e., $f(x) \in (a,b)$

Claim: $x \in U_q \setminus \overline{U_p}$ for suitable $p<q$

and $f(U_q \setminus \overline{U_p}) \subset (a,b)$.