Lect12-20180226

Feb 14, Wednesday, 2018 6:18 PM

More about Torus

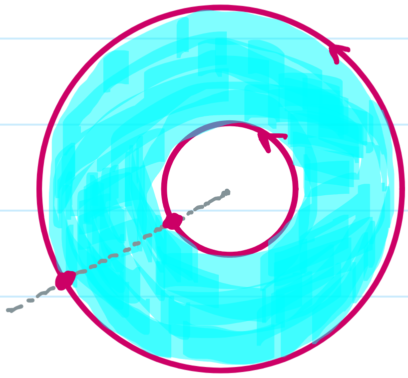
- Group version $\mathbb{R}^2/\mathbb{Z}^2$, i.e.,

  $(x_1, x_2) \sim (y_1, y_2)$ if $\begin{cases} x_1 - x_2 \in \mathbb{Z} \\ y_1 - y_2 \in \mathbb{Z} \end{cases}$

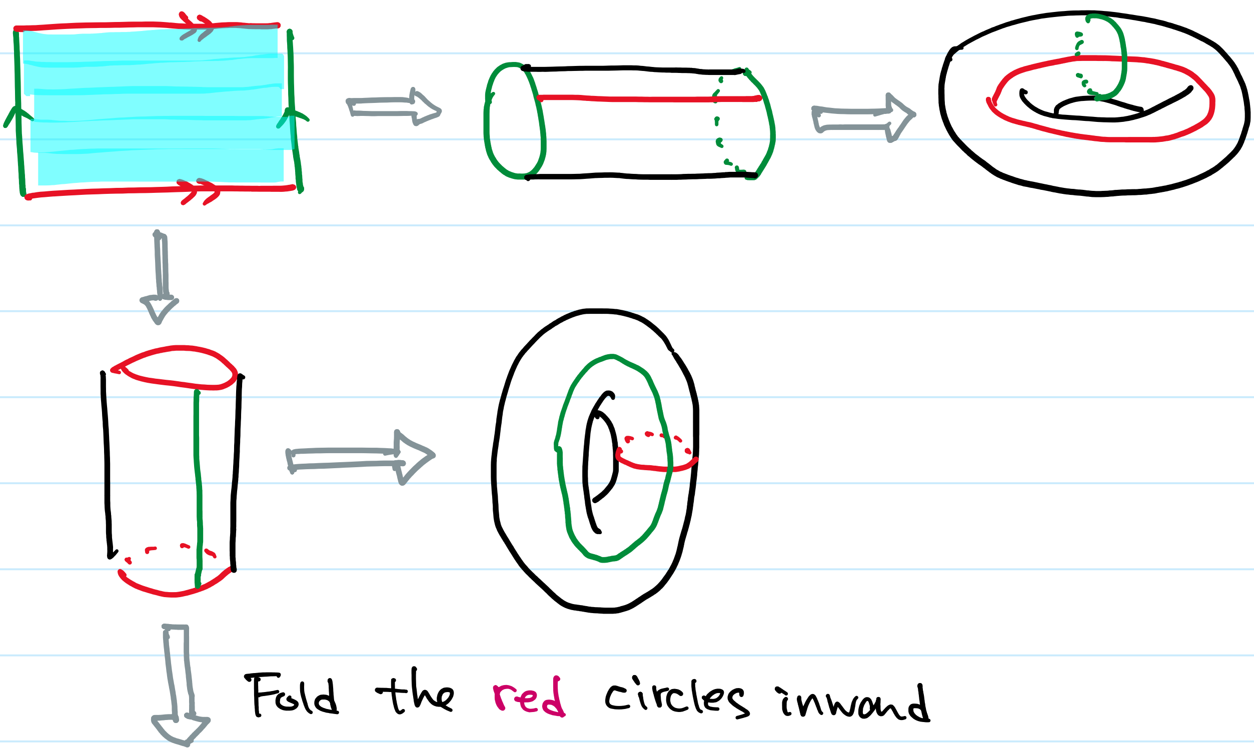
  $\Longleftrightarrow (x_1, x_2) - (y_1, y_2) \in \mathbb{Z}^2$

- Identify $ae^{i\theta}$ with $be^{i\theta}$ in

  $A = \{ z \in \mathbb{C} : a \leq |z| \leq b \}$



- Different "ways" to glue



Fold the red circles inward

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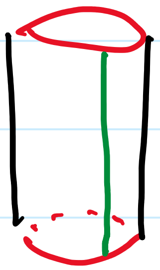
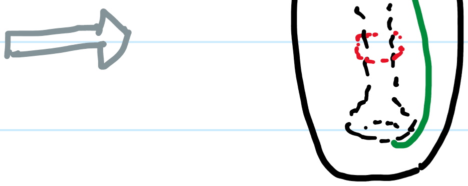
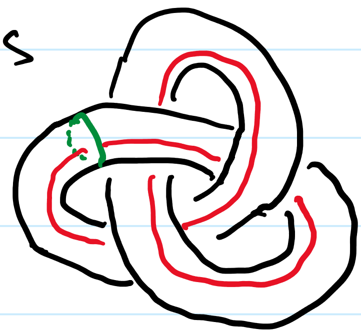


Fold the red circles inward



or even doing this



Facts. All these are the same torus.

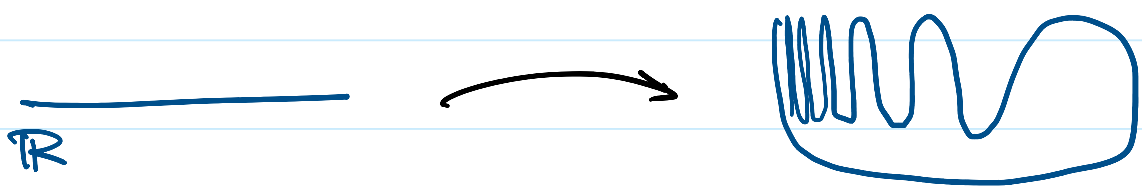
They are different "ways to put" the torus in $\mathbb{R}^3$.

An embedding of the torus in $\mathbb{R}^3$ is a mapping $h: S^1 \times S^1 \longrightarrow \mathbb{R}^3$ such that

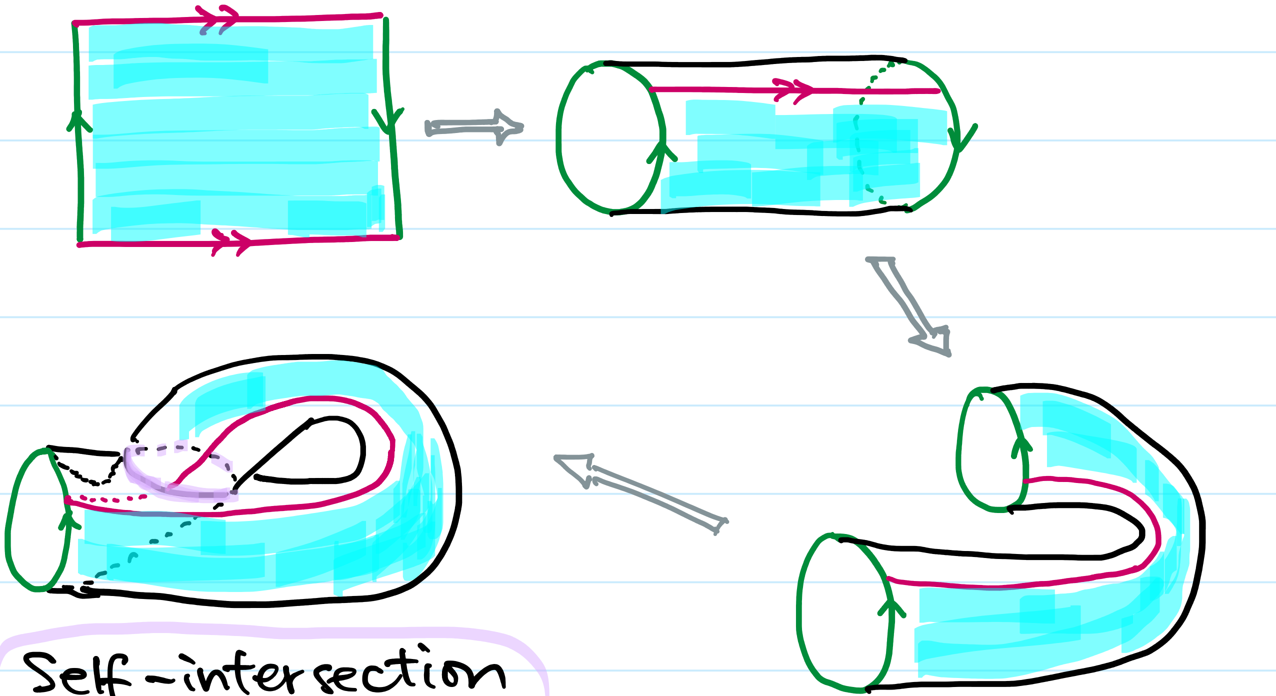
$$h: (S^1 \times S^1, \text{Std}) \longrightarrow (\underbrace{h(S^1 \times S^1), \text{subspace}}_{\text{any one of the above}})$$

is a homeomorphism

Example. Not an embedding of $\mathbb{R}$ into $\mathbb{R}^2$



Klein Bottle On $[0,1]\times[0,1]$, identify $(s,0)$ with $(s,1)$ and $(0,t)$ with $(1,1-t)$



Self-intersection occurs

Klein Bottle is **not** a subspace of $\mathbb{R}^3$

Digression about self-intersection

A "2-dimensional" subset $S \subset \mathbb{R}^3$ may be described by a function $f:\mathbb{R}^3 \to \mathbb{R}$,

$$\{(x_1,x_2,x_3)\in\mathbb{R}^3 : f(x_1,x_2,x_3)=0\}$$

In general, "k-dimensional" subset in $\mathbb{R}^n$ would be ......

A large group of k-dim subset in $\mathbb{R}^n$ can be described by

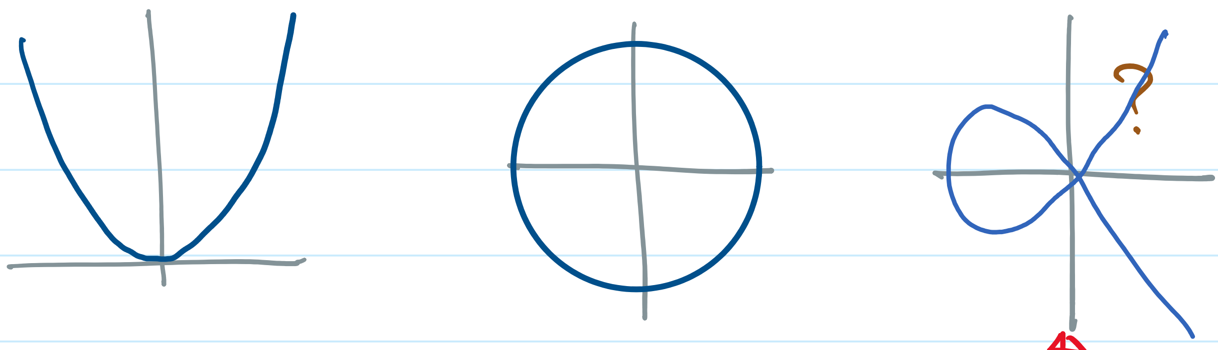
$$\{ x \in \mathbb{R}^n : f_j(x) = 0 \ , \ j = 1, \cdots, n-k \}$$

number of variables

number of equations.

Examples. In the case of $k=1$, $n=2$

$\{ y = x^2 \}$ $\{ x^2 + y^2 = 1 \}$ $\{ y^2 = x^3 + x \}$



No self-intersection

meets itself at certain points

Fact: not common non-generic

common generic situation

nowhere dense

dense

according to a natural topology on all mappings of $\mathbb{R}$ into $\mathbb{R}^2$

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What is the generic condition for

$$\underbrace{\{ x \in \mathbb{R}^n : f_j(x) = 0 , j = 1, \cdots, n-k \}}_{k\text{-dim object in } \mathbb{R}^n}$$

Answer: It has self-intersection of "dimension" $2k-n$.

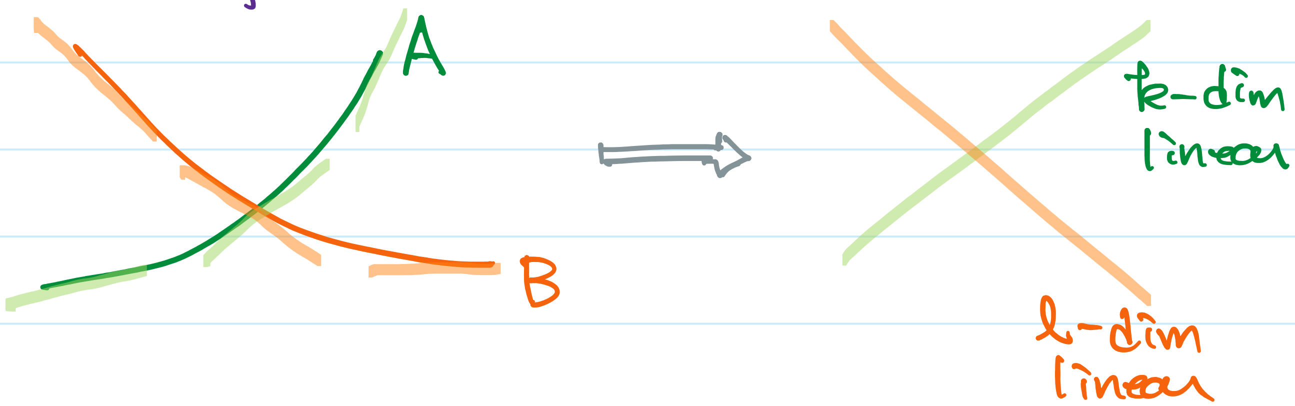
Application:

* A curve in $\mathbb{R}^2$, meets at points (dim=0)
* A curve in $\mathbb{R}^3$, no intersection (dim<0)
* A surface in $\mathbb{R}^3$, meets at curves (2+2-3)
* A surface in $\mathbb{R}^4$, meets at points
* A surface in $\mathbb{R}^5$, no intersection

Fact. Given two "objects" $A, B \subset \mathbb{R}^n$ where $\dim A = k$, $\dim B = l$

Generically, $\dim(A \cap B) = k + l - n$

Sketch of idea.



Lect12-p6

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Linear Algebra

What is the dimension of the intersection of a k-dim linear space with an l-dim linear space?

k-dim linear space is spanned by $v_1, v_2, \cdots, v_k \in \mathbb{R}^n$ as

$$\{ t_1 v_1 + t_2 v_2 + \cdots\cdots + t_k v_k : t_1, t_2, \cdots, t_k \in \mathbb{R} \}$$

And the l-dim one is

$$\{ s_1 w_1 + s_2 w_2 + \cdots + s_l w_l : s_1, \cdots, s_l \in \mathbb{R} \}$$

They have intersection $\Longleftrightarrow$

$$\exists \underbrace{t_1, t_2, \cdots, \cdots, t_k ; s_1, s_2, \cdots, s_l}_{\text{unknowns}} \in \mathbb{R}$$

such that

$$\sum_{j=1}^{k} t_k V_k = \sum_{i=1}^{l} s_i w_i$$

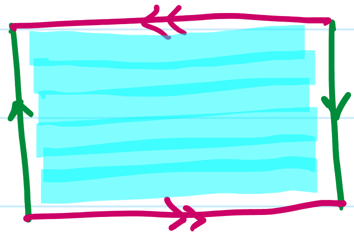
Equations on coordinates in $\mathbb{R}^n$

Result is a system of n equations with $k+l$ unknowns

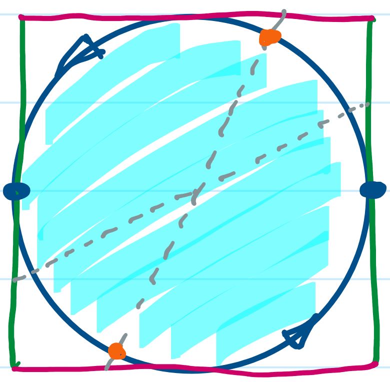
$\therefore$ Solution space has dimension $(k + l - n)$

Real Projective Plane, $\mathbb{RP}^2$

1. On $[0,1]\times[0,1]$, identify $(s,0)$ with $(1-s,1)$ and $(0,t)$ with $(1,1-t)$



2. On $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| \leq 1\}$, identify z with $-z$ whenever $|z|=1$, in S^1



3. The space of straight lines in $\mathbb{R}^3$ thru $\bar{0}$

Define $\sim$ on $\mathbb{R}^3 \setminus \{\bar{0}\}$ by

$\bar{x} \sim \bar{y}$ if $\underbrace{\exists\ 0 \neq \lambda \in \mathbb{R} \quad \bar{x} = \lambda \bar{y}}_{\bar{x},\,\bar{0},\,\bar{y} \text{ on the same st. line}}$

Then $[x] = \underbrace{\{ \lambda \bar{x} : 0 \neq \lambda \in \mathbb{R} \}}_{\text{basically, a st. line through } \bar{0}}$

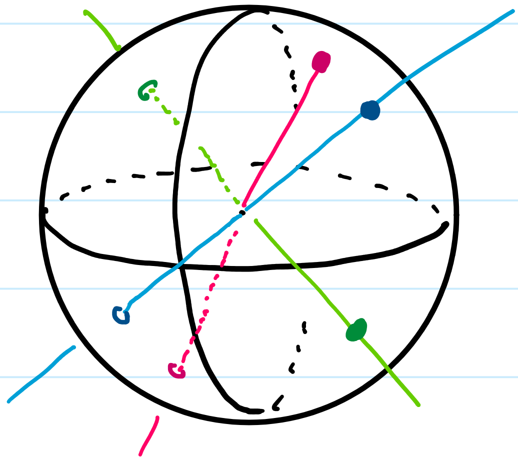
$$q : \mathbb{R}^3 \setminus \{\bar{0}\} \longrightarrow (\mathbb{R}^3 \setminus \{\bar{0}\}) / \sim$$

standard topology

quotient topology

How to visualize the open sets?

4. On $S^2 = \{ u \in \mathbb{R}^3 : \|u\| = 1 \}$, identify antipodal points, u with $-u$



Lect12-p9

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Exercise.

$$S^2/u\sim -u \xleftrightarrow{\text{homeomorphic}} (\mathbb{R}^3\setminus\{\bar{0}\})/x\sim\lambda y$$

$$[u] \longmapsto [u]$$

$$[u] = \{u,-u\} \qquad [u] = \{\lambda u : 0\neq\lambda\in\mathbb{R}\}$$

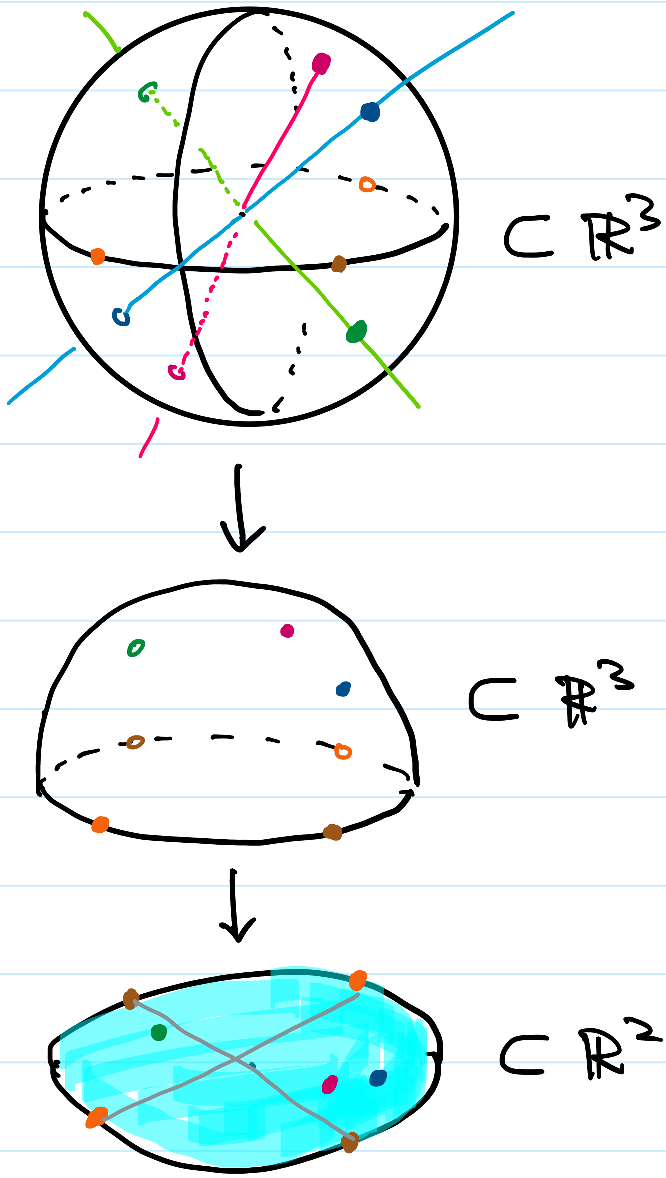
$$\left[\frac{x}{\|x\|}\right] \longmapsfrom [x]$$

Exercise

$$S^2/u\sim -u$$

homeomorphic

$$D^2/z\sim -z \text{ on } S^1$$



Real Projective Spaces $\mathbb{RP}^n$

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / (x \sim \lambda y,\ 0 \neq \lambda \in \mathbb{R})$$

$$= S^n / \text{antipodal}$$

Exercise. $\mathbb{RP}^1$ is homeomorphic to S^1

Note. $\mathbb{RP}^n \neq S^n$ for $n > 1$

Complex Projective Spaces $\mathbb{CP}^n$

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / (z \sim \mu w,\ 0 \neq \mu \in \mathbb{C})$$

$z, w \in \mathbb{C}^{n+1}$

$$= S^{2n+1} / (\xi \sim e^{i\theta} \xi)$$

$\{\xi \in \mathbb{C}^{n+1} : |\xi| = 1\}$

Exercise. $\mathbb{CP}^1 = S^2$

Matrix Quotient

Denote $O(n) = O_n(\mathbb{R}) = \{n\times n \text{ orthogonal matrices}\}$

$$= \{Q : Q^TQ = QQ^T = I\} \subset (\mathbb{R}^{n^2}, J_{std})$$

Note that $O(n-1) \hookrightarrow O(n)$ by

$$M \longmapsto \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & M \\ 0 & \end{bmatrix}$$

In group theory,

$$O(3)/O(2) = \{A\cdot O(2) : A \in O(3)\}$$

left Coset

Also, Quotient set of what ?

$$A \sim B \text{ if } \underbrace{A\cdot O(2) = B\cdot O(2)}$$

$$A^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & Q & \end{bmatrix}$$

$A^{-1}B(e_1) = ?$, $e_1 = (1,0,0)$

In this case, $A^{-1}B(e_1) = e_1$, i.e., $A(e_1) = B(e_1)$

S^2

unit length

$\therefore$ It is well-defined that

$$O(3)/O(2) \longrightarrow S^2$$
$$A \longmapsto A(e_1)$$

Fact. $O(3) \subset (\mathbb{R}^9, \mathcal{J}_{std})$ and $O(3)/O(2)$ with quotient topology

Then $O(3)/O(2) \longrightarrow S^2$ is homeomorphic

Similarly, $O(n+1)/O(n) = S^n$

Projective Space

As above, denote $(\pm O(2)) \subset O(3)$ as the set of all $\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & & \\ 0 & & Q \end{bmatrix}$, $Q \in O(2)$

Then $A \sim B$ if $A(\pm O(2)) = B(\pm O(2))$ and $A(e_1) = \pm B(e_1)$

$O(3)/(\pm O(2)) \longrightarrow \mathbb{R}P^2$ is homeomorphic