MATH4210: Financial Mathematics Tutorial 4

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Options

Recall from the previous tutorial that we have shown

Proposition

Show that the European put options with strike price K and maturity at time T satisfies $P_E(t, K) > Ke^{-r(T-t)} - S(t)$ for all $t < T$, where $S(t)$ is the stock price, r is the continuous compounding interest rate.

Also, one can always deduce that

Proposition

$P_E(t, K) > 0$ *for all* $t < T$

Question

Two vanilla put options are identical except for the maturity dates $T_1 < T_2$. If the interest rate is zero between T_1 and T_2, then $P_E(t, T_1) < P_E(t, T_2)$ at any time $t \leq T_1$.

Options

Answer

Suppose $P_E(t_0, T_1) \geq P_E(t_0, T_2)$ *for some* $t_0 \leq T$. *Make a portfolio: long one put with maturity* T_2 *and short one put with maturity* T_1 *at time* t_0. *Then, the values of the portfolio are*

$$
\begin{aligned}
\Pi(t_0) &= P_E(t_0, T_2) - P_E(t_0, T_1) \leq 0 \\
\Pi(T_1) &= P_E(T_1, T_2) - (K - S(T_1))^+ \\
&> \begin{cases} K - S(T_1) - (K - S(T_1)) & \text{if } S(T_1) < K \\ 0 - 0 & \text{if } S(T_1) \geq K \end{cases} \\
&= 0
\end{aligned}
$$

This is an arbitrage opportunity. Thus, we must have $P_E(t, T_1) < P_E(t, T_2)$ *at any time* $t \leq T_1$.

Options

Question

What happens if the continuous compounding interest rate is $r > 0$?

Answer

The original portfolio becomes

$$
\begin{aligned}
\Pi_0(t_0) &= P_E(t_0, T_2) - P_E(t_0, T_1) \\
\Pi_0(T_1) &= P_E(T_1, T_2) - (K - S(T_1))^+ \\
&> \begin{cases} K e^{-r(T_2 - T_1)} - S(T_1) - (K - S(T_1)) & \text{if } S(T_1) < K \\ 0 - 0 & \text{if } S(T_1) \geq K \end{cases}
\end{aligned}
$$

Answer

To obtain useful inequality, we make a new portfolio as follows: long one put option with maturity T_2, short $e^{-r(T_2-T_1)}$ put option with maturity T_1, and long $(1-e^{-r(T_2-T_1)})$ stock at time t_0. Then, the values of the portfolio are

$$\begin{aligned}
\Pi_1(t_0) &= P_E(t_0,T_2) - e^{-r(T_2-T_1)}P_E(t_0,T_1) + (1-e^{-r(T_2-T_1)})S(t_0) \\
\Pi_1(T_1) &= P_E(T_1,T_2) - e^{-r(T_2-T_1)}(K-S(T_1))^+ + (1-e^{-r(T_2-T_1)})S(T_1) \\
&> \begin{cases} Ke^{-r(T_2-T_1)} - S(T_1) \\ \quad -e^{-r(T_2-T_1)}(K-S(T_1)) + (1-e^{-r(T_2-T_1)})S(T_1) & \text{if } S(T_1) < K \\ 0 - 0 + (1-e^{-r(T_2-T_1)})K & \text{if } S(T_1) \geq K \end{cases} \\
&\geq 0
\end{aligned}$$

To satisfy no arbitrage opportunity assumption, we must have

$$e^{-r(T_2-T_1)}P_E(t,T_1) < P_E(t,T_2) + (1-e^{-r(T_2-T_1)})S(t)$$

Answer

Another approach: we make a new portfolio as follows: long one put option with maturity T_2, short one put option with maturity T_1, and long $(e^{-r(T_1-t_0)} - e^{-r(T_2-t_0)})K$ *cash at time t_0. Then, the values of the portfolio are*

$$\begin{aligned}
\Pi_1(t_0) &= P_E(t_0, T_2) - P_E(t_0, T_1) + (e^{-r(T_1-t_0)} - e^{-r(T_2-t_0)})K \\
\Pi_1(T_1) &= P_E(T_1, T_2) - (K - S(T_1))^+ + (1 - e^{-r(T_2-T_1)})K \\
&> \begin{cases} Ke^{-r(T_2-T_1)} - S(T_1) \\ \quad -(K - S(T_1)) + (1 - e^{-r(T_2-T_1)})K & \text{if } S(T_1) < K \\ 0 - 0 + (1 - e^{-r(T_2-T_1)})K & \text{if } S(T_1) \geq K \end{cases} \\
&\geq 0
\end{aligned}$$

To satisfy no arbitrage opportunity assumption, we must have

$$P_E(t, T_1) < P_E(t, T_2) + (e^{-r(T_1-t)} - e^{-r(T_2-t)})K$$

Options

Question

We have two inequalities now:

$$e^{-r(T_2-T_1)}P_E(t,T_1) < P_E(t,T_2) + (1-e^{-r(T_2-T_1)})S(t) \tag{1}$$

$$P_E(t,T_1) < P_E(t,T_2) + (e^{-r(T_1-t)} - e^{-r(T_2-t)})K. \tag{2}$$

Which one is tighter?

Answer

(2) is tighter because (2) + (3) implies (1):

$$P_E(t,T_2) > Ke^{-r(T_2-t)} - S(t) \tag{3}$$

(Simple algebra! Exercise !)

Options

Question

What happens if the underlying asset pays deterministic dividend D?

Proposition

(Put-Call Parity Relation with Dividend) Assume that the value of the dividends of the stock paid during $[t, T]$ is a deterministic constant D at time $t_D \in (t, T]$. Let $S(t)$ be the stock price, r be the continuous compounding interest rate , $C_E(t, K)$ and $P_E(t, K)$ be the prices of European call and put option at time t with strike K and maturity T respectively. We have

$$C_E(t, K) - P_E(t, K) = S(t) - Ke^{-r(T-t)} - De^{-r(t_D - t)}$$

Thus, we have

$$\begin{aligned} P_E(t,K) &= C_E(t,K) - S(t) + Ke^{-r(T-t)} + De^{-r(t_D-t)} \\ &> Ke^{-r(T-t)} + De^{-r(t_D-t)} - S(t) \end{aligned}$$

Answer

If the dividend is paid at time $t_D \in (t, T_1]$, it does not affect our above analysis. If the dividend is paid at time $t_D \in (T_1, T_2]$, the our portfolio becomes

$$\begin{aligned}
\Pi_1(t_0) &= P_E(t_0, T_2) - P_E(t_0, T_1) + (e^{-r(T_1 - t_0)} - e^{-r(T_2 - t_0)})K \\
\Pi_1(T_1) &= P_E(T_1, T_2) - (K - S(T_1))^+ + (1 - e^{-r(T_2 - T_1)})K \\
&> \begin{cases} Ke^{-r(T_2 - T_1)} + De^{-r(t_D - T_1)} - S(T_1) & \\ \quad -(K - S(T_1)) + (1 - e^{-r(T_2 - T_1)})K & \text{if } S(T_1) < K \\ 0 - 0 + (1 - e^{-r(T_2 - T_1)})K & \text{if } S(T_1) \geq K \end{cases} \\
&\geq 0
\end{aligned}$$

To satisfy no arbitrage opportunity assumption, we still have

$$P_E(t, T_1) < P_E(t, T_2) + (e^{-r(T_1 - t)} - e^{-r(T_2 - t)})K$$

Options

Question

Can it be tighter?

Answer

Yes, we make a new portfolio as follows: long one put option with maturity T_2, short one put option with maturity T_1, and long $((e^{-r(T_1-t_0)}-e^{-r(T_2-t_0)})K-De^{-r(t_D-t_0)})^+$ cash at time t_0. Then, the values of the portfolio are

$$\begin{aligned}
\Pi_1(t_0) &= P_E(t_0,T_2)-P_E(t_0,T_1)+((e^{-r(T_1-t_0)}-e^{-r(T_2-t_0)})K-De^{-r(t_D-t_0)})^+\\
\Pi_1(T_1) &= P_E(T_1,T_2)-(K-S(T_1))^+ +((1-e^{-r(T_2-T_1)})K-De^{-r(t_D-T_1)})^+\\
&> \begin{cases} Ke^{-r(T_2-T_1)}+De^{-r(t_D-T_1)}-S(T_1) & \\ \quad -(K-S(T_1))+(1-e^{-r(T_2-T_1)})K-De^{-r(t_D-T_1)} & \text{if } S(T_1)<K\\ 0-0+0 & \text{if } S(T_1)\geq K \end{cases}\\
&\geq 0
\end{aligned}$$

Answer

To satisfy no arbitrage opportunity assumption, we must have

$$P_E(t, T_1) < P_E(t, T_2) + ((e^{-r(T_1-t)} - e^{-r(T_2-t)})K - De^{-r(t_D-t)})^+$$

Question

Show that $S(t) - Ke^{-r(T-t)} < C_A(t) < S(t)$ *for any* t *before maturity* T

Answer

Note that $C_A(t) \geq C_E(t) > S(t) - Ke^{-r(T-t)}$. *To show* $C_A(t) < S(t)$, *suppose not, assume* $C_A(t_0) \geq S(t_0)$ *for some* $t_0 < T$. *We make a portfolio: long one stock and short an American call options. Then, if the option is never exercised, the values of the portfolio become:*

$$\Pi(t_0) = S(t_0) - C_A(t_0) \leq 0$$
$$\Pi(T) = S(T) \geq 0$$

If the option is exercised at time t_1, *then*

$$\Pi(T) = Ke^{r(T-t_1)} > 0$$

Note that for the first case, we also have $\mathbb{P}(\Pi(T) > 0) = \mathbb{P}(S(T) > 0) > 0$. *Thus, this is an arbitrage opportunity. Thus, we must have* $C_A(t) < S(t)$ *under no arbitrage opportunity assumption.*

Question

Show that $P_A(t) < K$ for all $t < T$ if the underlying asset pay deterministic dividends D at the some time $t_D \in (t, T]$.

Answer

Suppose $P_A(t_0) \geq K$ for some t_0. We construct a portfolio: short one American put option and long cash K. The values of the portfolio are:

$$\Pi(t_0) = K - P_A(t_0) \leq 0$$
$$\Pi(T) = Ke^{r(T-t_0)} > 0$$

if the option is never exercised.

Options

Answer

If the option is exercised at some $t_1 < t_D$*,*

$$\Pi(t_1) = Ke^{r(t_1 - t_0)} + S(t) - K = K(e^{r(t_1 - t_0)} - 1) + S(t)$$
$$\Pi(T) = K(e^{r(T - t_0)} - e^{r(T - t_1)}) + S(T) + De^{r(T - t_D)} > 0,$$

If the option is exercised at some $t_1 \geq t_D$*.*

$$\Pi(t_1) = Ke^{r(t_1 - t_0)} + S(t) - K = K(e^{r(t_1 - t_0)} - 1) + S(t)$$
$$\Pi(T) = K(e^{r(T - t_0)} - e^{r(T - t_1)}) + S(T) > 0,$$

These are arbitrage opportunity. Thus, we must have $P_A(t) < K$*.*

Question

Show that $C_E(t) \le C_A(t) < C_E(t) + De^{-r(t_D - t)}$ *if the underlying pay a deterministic dividend* D *at time* $t_D \in (t, T]$.

Answer

$C_E(t) \le C_A(t)$ *is trivial. We construct a portfolio: long one American call at* t *and exercise it at* $t_1 < t_D$. *Then, the values of the portfolio are*

$$\begin{aligned}
\Pi_1(t) &= C_A(t) \\
\Pi_1(t_1) &= S(t_1) - K \\
\Pi_1(t_D) &= S(t_D) + D - Ke^{r(t_D - t_1)} < S(t_D) - Ke^{-r(T - t_D)} + D
\end{aligned}$$

Answer

If we do not exercise the option, then we have

$$\Pi_1(t) = C_A(t)$$
$$\Pi_1(t_D) = C_A(t_D)$$

We construct another portfolio as follows: long one European call option and long $De^{-r(t_D-t)}$ cash. Then, the values of the portfolio are

$$\Pi_2(t) = C_E(t) + De^{-r(t_D-t)}$$
$$\Pi_2(t_D) = C_E(t_D) + D = C_A(t_D) + D > \Pi_1(t_D)$$

Thus, we have $\Pi_2(t) > \Pi_1(t)$, i.e. $C_A(t) < C_E(t) + De^{-r(t_D-t)}$ for all $t < t_D$.

Options

If we have the assumption that

$$D > K - e^{-r(T-t_D)}\min(K, S(T)),$$

one can deduce that $C_A(t) > C_E(t)$ for $t < t_D$ (exercise!). If $S(T) \to 0$, this assumption becomes unrealistic. In some cases, we have very strong confidence that $S(T)$ will have a lower bound $B \leq K$, e.g. the company owns a lot of government bond. Then, the assumption becomes more realistic:

$$D > K - e^{-r(T-t_D)}B$$

Example

Suppose $r = 0$, $S(t) = S(0)$ for $t \in [0, \tau)$, then at time τ, one has a dividend D, so that $S(t) = S(0) - D$ for $t \in [\tau, T]$. In this case, an American call will be different to an European call, since it is optimal to exercise the call option before the dividend time τ.