Models of 2-dimensional hyperbolic space and relations among them; Hyperbolic length, lines, and distances

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MATH4900E Presentation 2, 5th October 2020

Outline

Measurements in the Poincaré Disc Model (Hui)

Möbius transformations of $\mathbb{D}$

Hyperbolic length and distance in $\mathbb{D}$

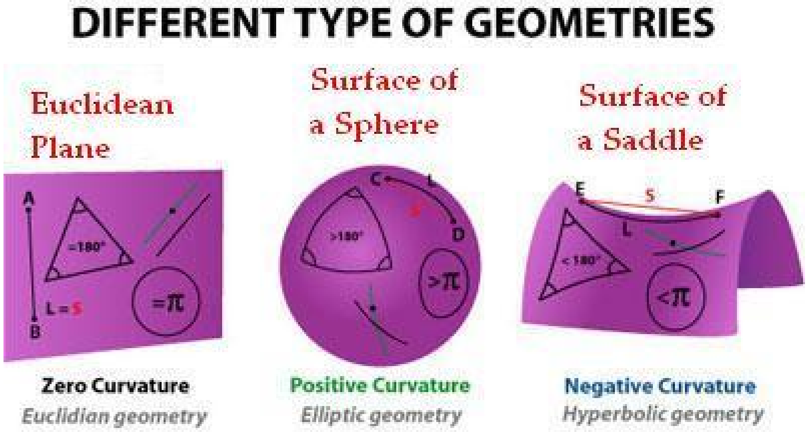
Conclusion

Boundary, Length, Orientation-preserving isometries, Geodesics and Angles

Reference

Upper half-plane model $\mathbb{H}$

Introduction to Upper half-plane model - continued



Hyperbolic geometry

Five Postulates of Hyperbolic geometry:

1. A straight line segment can be drawn joining any two points.

2. Any straight line segment can be extended indefinitely in a straight line.

3. A circle may be described with any given point as its center and any distance as its radius.

4. All right angles are congruent.

5. For any given line R and point P not on R, in the plane containing both line R and point P there are at least two distinct lines through P that do not intersect R.

Some interesting facts about hyperbolic geometry

1. Rectangles don't exist in hyperbolic geometry.

2. In hyperbolic geometry, all triangles have angle sum $< \pi$

3. In hyperbolic geometry if two triangles are similar, they are congruent.

4. Two triangles have the same area if and only if they have the same angle-sum.

Upper half-plane Model

Model: A choice of an underlying space and a choice of how to represent basic geometric objects. such as points and lines, in this underlying space.

The upper half-plane model

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$$

The upper half-plane model is not the only choice!
We have other options like:
$\mathbb{I}$, the Interior of the disk model (Hui)
$\mathbb{J}$, the Jemisphere model
$\mathbb{K}$, the Klein model
$\mathbb{L}$, the Hyperboloid model, etc...

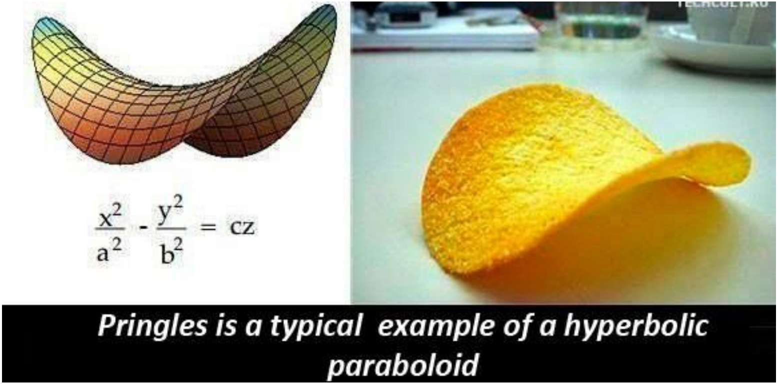
Introduction to Upper half-plane model

Upper half-plane model (or Poincaré half-plane model) is named after Henri Poincaré, but it originated with Eugenio Beltrami, who used it, along with the Klein model and the Poincaré disk model (due to Bernhard Riemann).

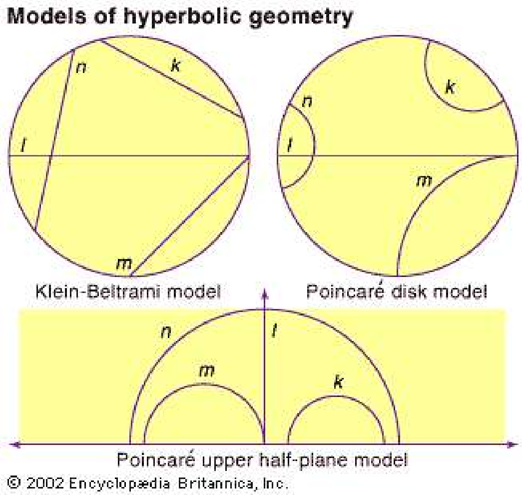
Hyperbolic plane geometry is also the geometry of saddle surfaces and pseudospherical surfaces (potato chips!), surfaces with a constant negative Gaussian curvature.

In the Poincaré half-plane model, the hyperbolic plane is flattened into a Euclidean half-plane. As part of the flattening, many of the lines in the hyperbolic plane appear curved in the model.

Introduction to Upper half-plane model-continued



Introduction to Upper half-plane model-continued



Upper half-plane Model

Definition 1.1

Two types of hyperbolic lines:
The intersection of $\mathbb{H}$ with a Euclidean line in $\mathbb{C}$ perpendicular to the real axis $\mathbb{R}$ in $\mathbb{C}$ or
The intersection of $\mathbb{H}$ with a Euclidean circle centred on the real axis $\mathbb{R}$

Hence we have the following proposition:

Proposition 1.2

For each pair of p and q of distinct points in $\mathbb{H}$, there exists a unique hyperbolic line l in $\mathbb{H}$ passing through p and q.

Upper half-plane Model

Proof of Propostion 1.2

Case 1: Consider the first type of hyperbolic lines in Definition 1.1: The intersection of $\mathbb{H}$ with a Euclidean line in $\mathbb{C}$ perpendicular to the real axis $\mathbb{R}$ in $\mathbb{C}$. Then, the Euclidean line L given by the equation $L = \{z \in \mathbb{C} : Re(z) = Re(p)\}$ is perpendicular to the real axis and passes through both p and q. So, the hyperbolic line l $=$ $\mathbb{H} \cap L$ is the desired hyperbolic line through p and q.

Upper half-plane Model

Proof of Propostion 1.2 - continued

Case 2: Consider the second type of hyperbolic lines in Definition 1.1: The intersection of $\mathbb{H}$ with a Euclidean circle centred on the real axis $\mathbb{R}$. Let's assume $Re(p) \neq Re(q)$. Since the Euclidean line through p and q is no longer perpendicular to $\mathbb{R}$, we need to construct a Euclidean circle centred on the real axis $\mathbb{R}$ that passes through p and q.
Let L_{pq} be the Euclidean line segment joining p and q and let K be the perpendicular bisector of L_{pq}. Then, every Euclidean circle that passes through p and q has its centre on K. Since p and q have non-equal real parts, the Euclidean line K is not parallel to $\mathbb{R}$, and so K and $\mathbb{R}$ intersect at a unique point c.

Upper half-plane Model

Proof of Propostion 1.2 - continued

Let A be the Euclidean circle centred at this point of intersection c with radius $|c - p|$, so that A passes through p. Since c lies on K, we have that $|c - p| = |c - q|$, and so A passes through q as well. The intersection $l = c \cap A$ is then the desired hyperbolic line passing through p and q.
The uniqueness of the hyperbolic line passing through p and q comes from the uniqueness of the Euclidean lines and Euclidean circles used in its construction. This completes the proof of Proposition 1.2.

Upper half-plane Model

Since we have chosen the underlying space $\mathbb{H}$ for this model of the hyperbolic plane to be contained in $\mathbb{C}$, and since we have chosen to define hyperbolic lines in $\mathbb{H}$ in terms of Euclidean lines and Euclidean circles in $\mathbb{C}$, we are able to use whatever facts about Euclidean lines and Euclidean circles we know to analyze the behaviour of hyperbolic lines.

Example

Let p and q be distinct points in $\mathbb{C}$ with non-equal real parts and let A be the Euclidean circle centred on $\mathbb{R}$ and passing through p and q. Express the Euclidean centre c and the Euclidean radius r of A in terms of Re(p), Im(p), Re(q), and Im(q).

Answer

Let L_{pq} be the Euclidean line segment joining p and q. The midpoint of L_{pq} is $\frac{1}{2}(p+q)$ and the slope of L_{pq} is $m = \frac{Im(q)-Im(p)}{Re(q)-Re(p)}$. The perpendicular bisector K of L_{pq} passes through $\frac{1}{2}(p+q)$ and has slope $-\frac{1}{m} = \frac{Re(p)-Re(q)}{Im(q)-Im(p)}$, and so K has the equation:

$$y - \frac{1}{2}(Im(p) + Im(q)) = [\frac{Re(p) - Re(q)}{Im(q) - Im(p)}](x - \frac{1}{2}(Re(p) + Re(q)))$$

Answer - continued

The Euclidean centre of c of A is the x-intercept of K, which is

$$c = [-\frac{1}{2}(Im(p) + Im(q))][\frac{Im(q) - Im(p)}{Re(p) - Re(q)}] + \frac{1}{2}(Re(p) + Re(q))$$

$$= \frac{1}{2}[\frac{Im(p)^2 - Im(q)^2 + Re(p)^2 - Re(q)^2}{Re(p) - Re(q)}]$$

$$= \frac{1}{2}[\frac{|p|^2 - |q|^2}{Re(p) - Re(q)}]$$

The Euclidean radius of A is

$$r = |c - p| = \left|\frac{1}{2}[\frac{|p|^2 - |q|^2}{Re(p) - Re(q)}] - p\right|$$

Upper half-plane Model

Definition 1.3

Two hyperbolic lines are parallel if they are disjoint.

In Euclidean geometry, parallel lines exist, and in fact, if L is a Euclidean line and if a is a point in $\mathbb{C}$ not on L, then there exists one and only one line K through a that is parallel to L.
In fact, in Euclidean geometry parallel lines are also equidistant, that is, if L and K are parallel Euclidean lines and if a and b are points on L, then the Euclidean distance from a to K is equal to the Euclidean distance from b to K.
In hyperbolic geometry, parallelism behaves much differently. Though we do not yet have a means of measuring hyperbolic distance, we can consider parallel hyperbolic lines qualitatively.

Upper half-plane Model

Theorem 1.4

Let l be a hyperbolic line in $\mathbb{H}$ and let p be a point in $\mathbb{H}$ not on l. Then, there exist infinitely many different hyperbolic lines through p that are parallel to l.

Proof of Theorem 1.4

As in the proof of Proposition 1.2, there are two cases to consider. First, suppose that l is contained in a Euclidean line L. Since p is not on L, there exists a Euclidean line K through p that is parallel to L. Since L is perpendicular to $\mathbb{R}$, we have that K is perpendicular to $\mathbb{R}$ as well. So, one hyperbolic line in $\mathbb{H}$ through p and parallel to l is the intersection $\mathbb{H} \cap K$.

Upper half-plane Model

Proof of Theorem 1.4 - continued

To construct another hyperbolic line through p and parallel to l, take a point x on $\mathbb{R}$ between K and L, and let A be the Euclidean circle centred on $\mathbb{R}$ that passes through x and p. We know that such a Euclidean circle A exists since Re(x) $\neq$ Re(p).

By construction, A is disjoint from L , and so the hyperbolic line $\mathbb{H}$ $\cap A$ is disjoint from l . That is, $\mathbb{H} \cap$ A is a second hyperbolic line through p that is parallel to f. Since there are infinitely many points on $\mathbb{R}$ between K and L, this construction gives infinitely many different hyperbolic lines through p and parallel to l.

Upper half-plane Model

Now, suppose that l is contained in a Euclidean circle A. Let D be the Euclidean circle that is concentric to A and that passes through p. Since concentric circles are disjoint and have the same centre, one hyperbolic line through p and parallel to l is the intersection $\mathbb{H} \cap D$.

To construct a second hyperbolic line through p and parallel to l, take any point x on $\mathbb{R}$ between A and D. Let E be the Euclidean circle centred on $\mathbb{R}$ that passes through x and p. Again by construction, E and A are disjoint, and so $\mathbb{H} \cap E$ is a hyperbolic line through p parallel to l.

As above, since there are infinitely many points on $\mathbb{R}$ between A and D, there are infinitely many hyperbolic lines through p parallel to l.

Upper half-plane Model

Example

Give an explicit description of two hyperbolic lines in $\mathbb{H}$ through i and parallel to the hyperbolic line $l = \mathbb{H} \cap \{z \in \mathbb{C} | Re(z) = 3\}$.

Answer

One hyperbolic line through i that is parallel to l is the positive imaginary axis $l = \mathbb{H} \cap \{Re(z) = 0\}$. To get a second hyperbolic line through i and parallel to l, take any point x on $\mathbb{R}$ between 0 and 3, say x $=$ 2, and consider the Euclidean circle centred on $\mathbb{R}$ through 2 and i.
By last example, the Euclidean centre c of A is c $= \frac{3}{4}$ and the Euclidean radius of A is r $= \frac{5}{4}$. Since the real part of every point on A is at most 2, the hyperbolic line $\mathbb{H} \cap C$ is a hyperbolic line passing through i that is parallel to l.

Upper half-plane Model

Example

Give an explicit description of two hyperbolic lines in $\mathbb{H}$ through i and parallel to the hyperbolic line l $= \mathbb{H} \cap A$, where A is the Euclidean circle with Euclidean centre -2 and Euclidean radius 1.

Answer

The Euclidean circle D through i and concentric to A has Euclidean centre -2 and Euclidean radius $\sqrt{5} = |i - (-2)|$, and so one hyperbolic line through i parallel to A is $\mathbb{H} \cap D$.

Upper half-plane Model

Answer - continued

To construct a second hyperbolic line through i and parallel to l, start by taking a point x on $\mathbb{R}$ between A and D, say x = -4. Let E be the Euclidean circle centred on $\mathbb{R}$ passing through -4 and i. By example before, the Euclidean centre c of E is $c = -\frac{15}{8}$ and the Euclidean radius is $r = \frac{17}{8}$.

It is easy to see that the two Euclidean circles $\{|z + 2 = 1|\}$ and $\{\left|z + \frac{15}{8}\right| = \frac{17}{8}\}$ are disjoint, and so the hyperbolic line $\mathbb{H} \cap E$ is a hyperbolic line passing through i that is parallel to l.

Stereographic Projection

Let $\mathbb{S}$ be the unit circle in $\mathbb{C}$.
Let ξ be a function such that:
Given a point z in $\mathbb{S}^1$ - $\{i\}$, let K_z be the Euclidean line passing through i and z, and set $\xi(z) = \mathbb{R} \cap K_z$. This function is well-defined, since K_z and $\mathbb{R}$ intersect in a unique point as long as Im(z) $\neq$ 1.

This operation is referred to as stereographic projection.

Slope = m =

$$\frac{Im(z) - Im(i)}{Re(z) - Re(i)} = \frac{Im(z) - 1}{Re(z)}$$

y-intercept =

$$Im(i) = 1$$

Stereographic Projection

Hence, the equation of K_z is

$$y - 1 = \frac{Im(z) - 1}{Re(z)} x$$

To find the x-intercept of K_z, we set y $= 0$, and we have:

$$-1 = \frac{Im(z) - 1}{Re(z)} x$$

$$x = \frac{Re(z)}{1 - Im(z)}$$

An explicit formula for $\xi^{-1} : \mathbb{R} \to \mathbb{S}^1 - \{i\}$

If c = 0, then the Euclidean line L_c passing through i and 0 intersects $\mathbb{S}^1$ at $\pm i$, and so $\xi^{-1}(0) = -i$. Given a point $c \neq 0$ in $\mathbb{R}$, the equation of the Euclidean line L_c passing through c and i is

$$y = -\frac{1}{c}(x - c) = -\frac{1}{c}x + 1$$

To find where L_c intersects $\mathbb{S}^1$, we find the values of x for which

$$|x + iy| = \left|x + i(-\frac{1}{c}x + 1)\right| = 1$$

An explicit formula for $\xi^{-1} : \mathbb{R} \to \mathbb{S}^1 - \{i\}$ - continued

which simplifies to

$$x[(1 + \frac{1}{c^2})x - \frac{2}{c}] = 0$$

Since x = 0 corresponds to i, we have that

$$x = \frac{2c}{c^2 + 1}$$

So,

$$\xi^{-1}(c) = \frac{2c}{c^2 + 1} + i\frac{1 - c^2}{c^2 + 1}$$

Stereographic Projection

In fact, ξ is a bijection between $\mathbb{S}^1 - \{i\}$ and $\mathbb{R}$. Geometrically, this follows from the fact that a pair of distinct points in $\mathbb{C}$ determines a unique Euclidean line. If z and w are points of $\mathbb{S}^1 - \{i\}$ for which $\xi(z) = \xi(w)$ then K_z and K_w both pass through the same point of $\mathbb{R}$, namely $\xi(z) = \xi(w)$. However, since both K_z and K_w pass through i as well, this forces the two lines K_z and K_w to be equal, and so z = w.

Since we obtain $\mathbb{R}$ from $\mathbb{S}^1$ by removing a single point of $\mathbb{S}^1$, namely i, we can think of constructing the Euclidean circle $\mathbb{S}^1$ by starting with the Euclidean line $\mathbb{R}$ and adding a single point.

The Riemann Sphere $\overline{\mathbb{C}}$

Motivated by this, one possibility for a space that contains $\mathbb{H}$ and in which the two seemingly different types of hyperbolic line are unified is the space that is obtained from $\mathbb{C}$ by adding a single point. This is the classical construction from Complex Analysis of the Riemann sphere $\overline{\mathbb{C}}$.

Definition

The Riemann sphere is the union $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

To visualise Riemann Sphere:

https://www.youtube.com/watch?v=l3nlXJHD714

The Riemann Sphere $\overline{\mathbb{C}}$

Definition

A set X in $\mathbb{C}$ is open if for each z $\in$ X, there exists some $\varepsilon > 0$ so that $U_\varepsilon(z) \subset X$, where $U_\varepsilon(z) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$ is the Euclidean disc of radius ε centred at z.

A set X in $\mathbb{C}$ is closed if its complement $\mathbb{C}$ - X in $\mathbb{C}$ is open.

A set X in $\mathbb{C}$ is bounded if there exists some constant $\varepsilon > 0$ so that $X \subset U_\varepsilon(0)$.

The Riemann Sphere $\overline{\mathbb{C}}$

In order to extend this definition to $\overline{\mathbb{C}}$, we need only define what $U_\varepsilon(z)$ means for each point z of $\mathbb{C}$ and each $\varepsilon > 0$. Since all but one point of $\overline{\mathbb{C}}$ lies in $\mathbb{C}$, it makes sense to use the definition we had above wherever possible, and so for each point z of $\mathbb{C}$ we define

$$U_\varepsilon(z) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$$

It remains only to define $U_\varepsilon(\infty)$, which we take to be

$$U_\varepsilon(\infty) = \{w \in \mathbb{C} : |w| > \varepsilon\} U \{\infty\}$$

The Riemann Sphere $\overline{\mathbb{C}}$

If D is an open set in $\mathbb{C}$, then D is also open in $\overline{\mathbb{C}}$.
For example, $\mathbb{H}$ is an open subset of $\mathbb{C}$, so $\mathbb{H}$ is an open subset of $\overline{\mathbb{C}}$.

The set $E = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$ is open in $\overline{\mathbb{C}}$. We need to show that for each point z of E, there is some $\varepsilon > 0$ so that $U_\varepsilon(z) \subset E$. Since $E = U_1(\infty)$, we can find a suitable ε for z $= \infty$, namely $\varepsilon = 1$. For a point z of E - $\{\infty\}$, note that the Euclidean distance from z to $\partial E = \mathbb{S}^1$ is $|z| - 1$, and so we have that $U_\varepsilon(z) \subset E$ for any $0 < \varepsilon < |z| - 1$.

On the other hand, the unit circle $\mathbb{S}^1$ in $\mathbb{C}$ is not open. No matter which point z of $\mathbb{S}^1$ and which $\varepsilon > 0$ we consider, we have that $U_\varepsilon(z)$ does not lie in $\mathbb{S}^1$, as $U_\varepsilon(z)$ necessarily contains the point $(1 + \frac{1}{2}\varepsilon)z$ whose modulus is
$\left|(1 + \frac{1}{2}\varepsilon)z\right| = (1 + \frac{1}{2}\varepsilon)|z| = 1 + \frac{1}{2}\varepsilon > 1.$

The Riemann Sphere $\overline{\mathbb{C}}$

Defintion

A set X in $\overline{\mathbb{C}}$ is closed if its complement $\overline{\mathbb{C}}$ - X in $\overline{\mathbb{C}}$ is open.

Example

The unit circle $\mathbb{S}^1$ is closed in $\mathbb{C}$, since its complement is the union

$$\overline{\mathbb{C}} - \mathbb{S}^1 = U_1(0) \cup U_1(\infty)$$

Definition

A sequence $\{z_n\}$ of points in $\overline{\mathbb{C}}$ converges to a point z of $\overline{\mathbb{C}}$ if for each $\varepsilon > 0$, there exists N so that $Z_n \in U_\varepsilon(z)$, $\forall n > N$.

The Riemann Sphere $\overline{\mathbb{C}}$

Examples

$\{z_n = \frac{1}{n} | n \in \mathbb{N}\}$ converges to 0 in $\overline{\mathbb{C}}$, and that $\{W_n = n | n \in \mathbb{N}\}$ converges to ∞ in $\overline{\mathbb{C}}$.

Definition

A circle in $\overline{\mathbb{C}}$ is either a Euclidean circle in $\mathbb{C}$, or the union of a Euclidean line in $\mathbb{C}$ with $\{\infty\}$.

Definition

A function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is continuous at $z \in \overline{\mathbb{C}}$ if for each $\varepsilon > 0$, there exists $\delta > 0$ so that $w \in U_\delta(z)$ implies that $f(w) \in U_\varepsilon(f(z))$. A function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is continuous if it is continuous at every point z of $\overline{\mathbb{C}}$.

The Riemann Sphere $\overline{\mathbb{C}}$

Examples

Constant functions, sums, differences, products, quotients and compositions of continuous functions (when they are defined) from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ are continuous.

Definition

A function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism if f is a bijection and if both f and f^{-1} are continuous.

Example

The function $J : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by

$$J(z) = \frac{1}{z}, z \in \mathbb{C} - \{0\}, J(0) = \infty, J(\infty) = 0$$

is a homeomorphism of $\overline{\mathbb{C}}$.

The Riemann Sphere $\overline{\mathbb{C}}$

Solution

J is continuous on $\overline{\mathbb{C}}$.

$$\lim_{z \to 0} J(z) = \lim_{z \to 0} \frac{1}{z} = \infty = J(0)$$

$$\lim_{z \to \infty} J(z) = \lim_{z \to \infty} \frac{1}{z} = 0 = J(\infty)$$

J is subjective, since J(J(z)) = z.

J is injective. Let z, w $\in \overline{\mathbb{C}}$. Suppose J(z) = J(w), then $\frac{1}{z} = \frac{1}{w}$, so z = w.

Since J is bijective, J^{-1} exists. Note that $J^{-1}(z) = J(z)$, J^{-1} is also continuous and bijective.

The Riemann Sphere $\overline{\mathbb{C}}$

Definition

Homeo($\overline{\mathbb{C}}$) $= \{f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} |$ f is a homeomorphism$\}$

The inverse of a homeomorphism is again a homeomorphism.

Also, the composition of two homeomorphisms is again a homeomorphism, since the composition of bijections is again a bijection and since the composition of continuous functions is again continuous.

As the identity homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by f(z) $=$ z is a homeomorphism, we have that Homeo($\overline{\mathbb{C}}$) is a group.

Poincaré Disc Model $\mathbb{D}$

Basic properties of Poincaré Disc Model

Jules Henri Poincaré (1881)

He contributed to algebraic topology, Algebraic geometry, number theory, etc.

A famous mathematician who formulated the Poincaré conjecture.

Basic properties of Poincaré Disc Model

Eugenio Beltrami

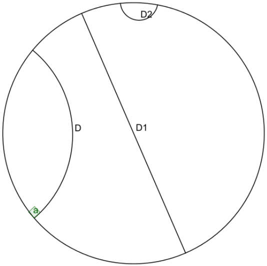
He used the Klein model and the Poincaré half-space model to propose Poincaré disc model.

But because of the rediscovery of Poincaré's work, it became more famous then Beltrami's work.

Basic properties of Poincaré Disc Model

Poincaré Disc Model (the conformal disc model)

It is a model of 2-d hyperbolic geometry in which the points of the geometry are inside the unit disc, and the straight lines consist of all circular arcs contained within that disc that are orthogonal to the boundary of the disc, and all diameters of the disc



Basic properties of Poincaré Disc Model

Definition

The underlying space is the open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

in the complex plane $\mathbb{C}$.

The unit circle $\mathbb{S}^1 = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ is called the circle at ∞ or boundary of $\mathbb{D}$ and its centre is at the origin of the Euclidean plane.

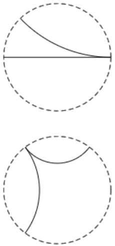
Basic properties of Poincaré Disc Model

Definition

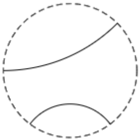
The hyperbolic line in $\mathbb{D}$ is the diameter or the arc that is orthogonal to $\mathbb{S}^1$. A point on $\mathbb{S}^1$ is called point at infinity.

1. Two hyperbolic lines are parallel if those lines share one point at infinity.
2. Two hyperbolic lines are parallel if those lines do not intersect.

- Asymptotically parallel



- Disjointly parallel



Basic properties of Poincaré Disc Model

Definition

A hyperbolic circle in $\mathbb{D}$ is a set in $\mathbb{D}$ of the form

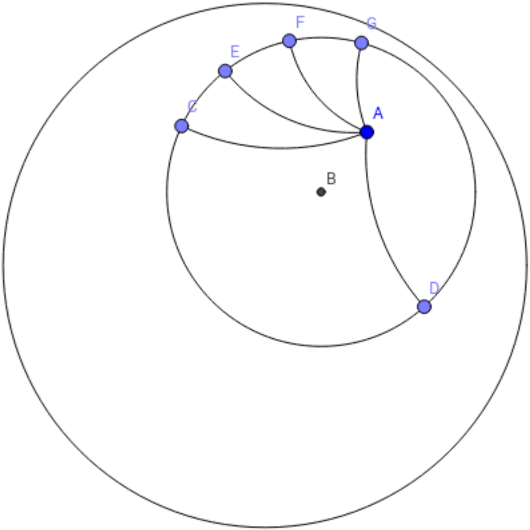
$$C = \{y \in \mathbb{D} \mid d_{\mathbb{D}}(x, y) = s\},$$

where $x \in \mathbb{D}$ and $s > 0$ are fixed.
x is the hyperbolic centre of C and s is the hyperbolic radius of C.

Properties of hyperbolic circle

- Hyperbolic circle is a Euclidean circle in $\mathbb{D}$.
- Hyperbolic centre is not Euclidean centre.
- Hyperbolic radii are not Euclidean radii.

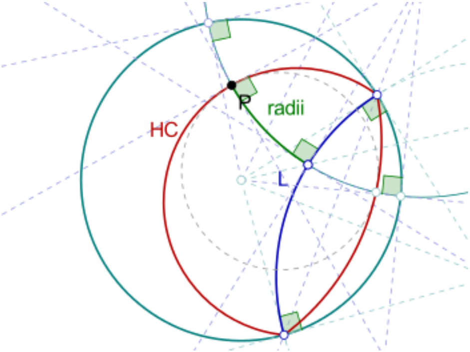
Basic properties of Poincaré Disc Model



Basic properties of Poincaré Disc Model

Definition

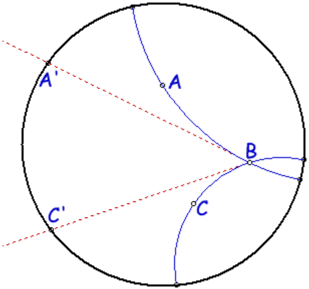
A hypercycle is a Euclidean circle arc or chord of the boundary circle that intersects the boundary circle at a non-right angle. Its axis is the hyperbolic line that shares the same two points at infinity.



Basic properties of Poincaré Disc Model

Angle in $\mathbb{D}$

- Equal to the measure of Euclidean angle.
- Is constructed by tangents of two arcs.



Basic properties of Poincaré Disc Model

Advantages

- is bounded by the circle at ∞.
- lives in the plane without need for a third dimension.

Disadvantages

- cannot use stragiht lines to model geodesics. (Klein model)
- cannot use real 2×2 matrices to describe isometric transformations. (Upper half-plane model)

Basic properties of Poincaré Disc Model

Construct the hyperbolic line that lies inside the disk.

Suppose the hyperbolic line through points P and Q is not the diameter of boundary circle

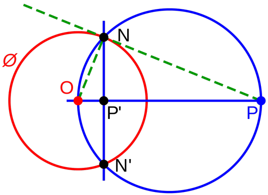
1. Let P^{-1} and Q^{-1} be the inversions in the boundary circle of point P and Q respectively.
2. Let M and N be the mid-points of segment PP^{-1} and QQ^{-1} respectively.
3. Draw line m through M perpendicular to segment PP^{-1}.
4. Draw line n through N perpendicular to segment QQ^{-1}.
5. Let C be where line m and line n intersect and draw circle c with centre C and passing through P (and Q).

The part of circle c that is inside the disk is the hyperbolic line.

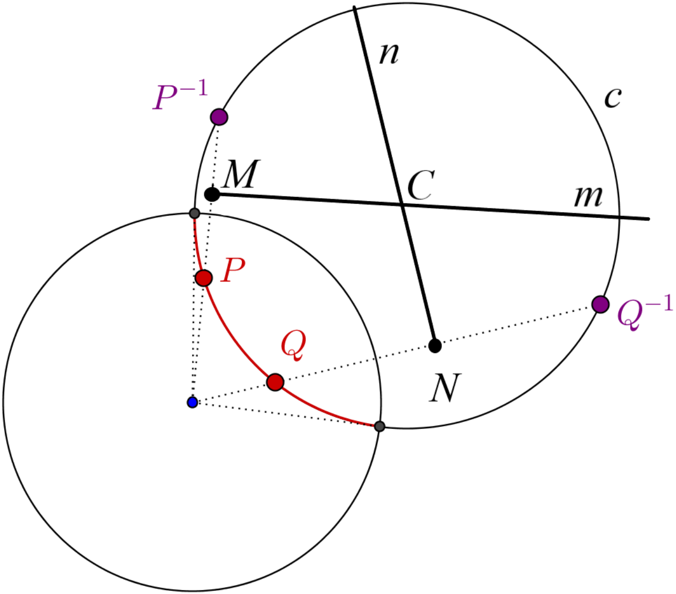
Basic properties of Poincaré Disc Model

Construct an inversion of a point P' that is inside the boundary circle C with centre O.

1. Draw ray r from O through P'.
2. Draw line s through P' perpendicular to r.
3. Let N be one of the points where O and s intersect.
4. Draw the segment ON.
5. Draw line t through N perpendicular to ON.
6. P is where ray r and line t intersect.



Basic properties of Poincaré Disc Model

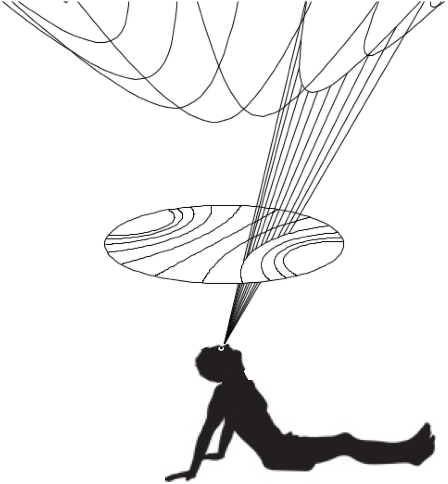


Basic properties of Poincaré Disc Model

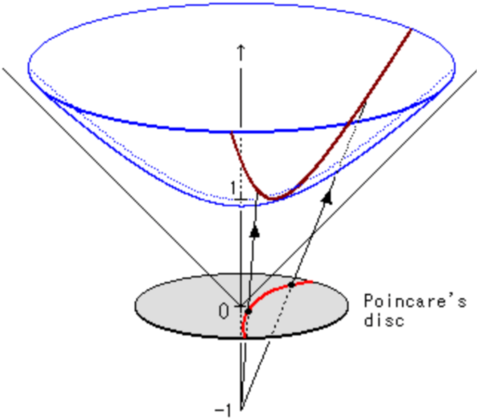
Another way to draw the hyperbolic line:

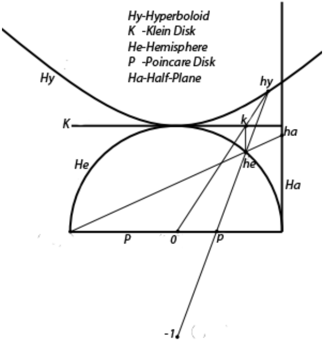
1. Let M be the mid-point of segment PQ and draw line m through M perpendicular to segment PQ.
2. Let P^{-1} be the inversion in the boundary circle of point P.
3. Let N be the mid-point of segment PP^{-1} and draw line n through N perpendicular to segment PP^{-1}.
4. Let C be where line m and n intersect and draw circle c with centre C and passing through P (and Q).

Basic properties of Poincaré Disc Model



Relation between $\mathbb{D}$ and other models



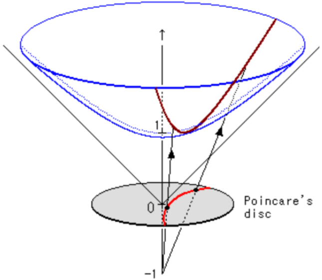


Relation between $\mathbb{D}$ and other models

Hyperboloid model

Poincaré disc model is related to the hyperboloid model projectively.

If a point $[t, x_1, ..., x_n]$ is on the upper sheet of the hyperboloid of the hyperboloid model, we can project it onto the plane $t = 0$ by intersecting it with a line drawn through [-1, 0, ..., 0].



Relation between $\mathbb{D}$ and other models

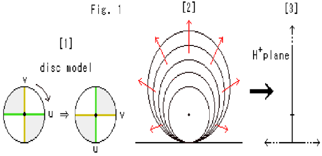
Upper half-plane model

Following figures will show that the change from Poincaré disc model to upper half-plane model.

(1) Turn the disk clockwise 90° as shown.

(2) Fix the centre and bottom, then expand the disk infinitely.

(3) The lower part of circumference becomes a horizontal straight line and other part goes to infinity.



Relation between $\mathbb{D}$ and other models

Upper half-plane model

Following figures will show the correlation between Poincaré disc model and upper half-plane model.

u: horizontal axis of the disk model.
v: vertical axis of the disk model.

1: u-axis $\Rightarrow$ Y-axis

2: positive end point of u-axis $\Rightarrow$ on X-axis

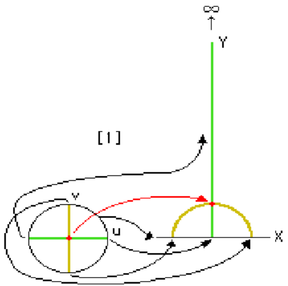
3: Negative end point of u-axis $\Rightarrow$ ∞

4: v-axis $\Rightarrow$ semicircle on the X-axis

5: Centre of the disk $\Rightarrow$ on semicircle

6: Circumference of disk $\Rightarrow$ X-axis

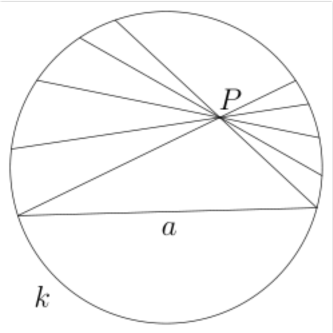
Fig. 2



Relation between $\mathbb{D}$ and other models

Klein disc model (projective model)

It is a model of hyperbolic geometry in which points are represented by the poins in the interior of the unit disc and lines are represented by the chords, straight line segments with ideal endpoints on the boundary sphere.



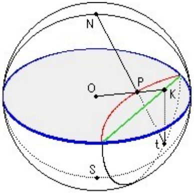
Relation between $\mathbb{D}$ and other models

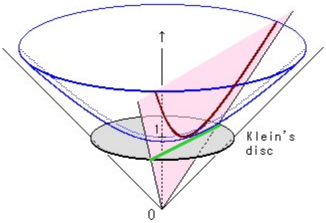
Klein disc model (projective model)

Both are the models that project the whole hyperbolic plane in a disc.

But $\mathbb{K}$ is an orthographic projection to hemisphere model and $\mathbb{D}$ is stereographic projection.

Also, the hyperbolic line in $\mathbb{K}$ is straight line/ chord of the circle, and the hyperbolic line in $\mathbb{D}$ is a diameter or an arc of the circle.





Length and distance in the upper half-plane model

Length and distance in the upper half-plane model

Recall that: The upper half-plane $\mathbb{C}$ is the set of complex numbers z with positive imaginary part: $\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$

Definition

The circle at infinity or boundary of $\mathbb{H}$ is defined to be the set $\partial\mathbb{H} = \{z \in \mathbb{C} | Im(z) = 0\} \cup \{\infty\}$. That is, $\partial\mathbb{H}$ is the real axis together with the point ∞.

Remark: We will use the conventions that, if a $\in \mathbb{R}$ and a $\neq 0$ then a$/\infty = 0$ and a$/0 = \infty$, and if b $\in \mathbb{R}$ then b$+\infty = \infty$. We leave $0/\infty$,$\infty/0$,∞/∞, $0/0$,$\infty \pm \infty$ undefined.

Path Integrals

Before we can define distances in $\mathbb{H}$ we need to recall how to calculate path integrals in $\mathbb{C}$ (equivalently, in $\mathbb{R}^2$).

By a path σ in the complex plane $\mathbb{C}$, we mean the image of a continuous function : [a, b] $\to \mathbb{C}$, where [a, b] $\subset \mathbb{R}$ is an interval. We will assume that σ is differentiable and that the derivative σ' is continuous. Thus a path is, heuristically, the result of taking a pen and drawing a curve in the plane. We call the points $\sigma(a)$, $\sigma(b)$ the end-points of the path σ. We say that a function σ : [a, b] $\to$ $\mathbb{C}$ whose image is a given path is a parametrisation of that path. Notice that a path will have lots of different parametrisations.

Path Integrals - continued

Example

Define $\sigma_1 : [0, 1] \to \mathbb{C}$ by $\sigma_1(t) = t + it$ and define $\sigma_2 : [0, 1] \to \mathbb{C}$ by $\sigma_2(t) = t^2 + it^2$. Then σ_1 and σ_2 are different parametrisations of the same path in $\mathbb{C}$, namely the straight (Euclidean) line from the origin to 1 + i.

Theorem

Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous function. Then the integral of f along a path σ is defined to be:

$$\int_\sigma f = \int_a^b f(\sigma(t))\big|\sigma'(t)\big|dt$$

here $|\cdot|$ denotes the usual modulus of a complex number, in this case,

$$\big|\sigma'(t)\big| = \sqrt{(Re(\sigma'(t))^2 + Im(\sigma'(t))^2}$$

Path Integrals - continued

To calculate the integral of f along the path σ we have to choose a parametrisation of that path. Any two parametrisations of a given path will always give the same answer. For this reason, we shall sometimes identify a path with its parametrisation.

Example

Consider the two parametrisations:

$$\sigma_1 : [0, 2] \to \mathbb{H} : t \mapsto t + i$$

$$\sigma_2 : [1, 2] \to \mathbb{H} : t \mapsto (t^2 - t) + i$$

Example - continued

$\sigma_1'(t) = 1, \; Im(\sigma_1(t)) = 1.$

$$\int_{\sigma_1} f = \int_0^2 f(\sigma_1(t))\big|\sigma_1'(t)\big| dt = 2$$

$\sigma_2'(t) = 2t - 1, \; Im(\sigma_2(t)) = 1.$

$$\int_{\sigma_2} f = \int_1^2 f(\sigma_2(t))\big|\sigma_2'(t)\big| dt = \int_1^2 2t - 1 dt$$

$$= [t^2 - t]_1^2 = (4 - 2) - (1 - 1) = 2$$

In this example, we can see computing the path integral using the second parametrisation was harder than using the first parametrisation. So the choice of parametrisation is important.

Extension of path integral

Definition

A path σ with parametrisation $\sigma : [a, b] \to \mathbb{C}$ is piecewise continuously differentiable if there exists a partition $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ of [a, b] such that $\sigma : [a, b] \to \mathbb{C}$ is a continuous function and, for each j, $0 \leq j \leq n-1$, $\sigma : (t_j, t_{j+1}) \to \mathbb{C}$ is differentiable and has continuous derivative.

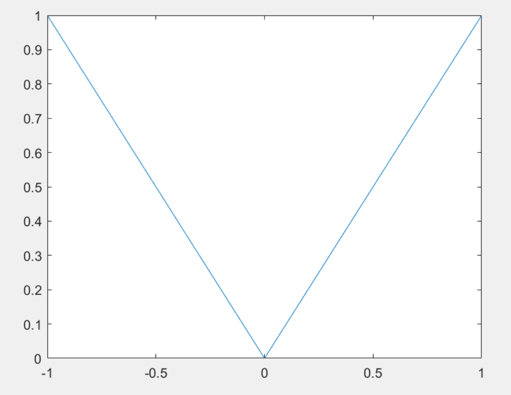
Roughly speaking this means that we allow the possibility that the path σ has finitely many 'corners'.

To define $\int_\sigma f$ for a piecewise continuously differentiable path σ we merely write σ as a finite union of differentiable sub-paths, calculating the integrals along each of these subpaths, and then summing the resulting integrals.

Extension of path integral - continued

Example

The path $\sigma(t) = (t, |t|), -1 \leq t \leq 1$ is piecewise continuously differentiable: it is differentiable everywhere except at the origin, where it has a 'corner'.



Distance in hyperbolic geometry

Metric

The metric of the model on the half-plane, $\{(x, y)|y > 0)\}$

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

Distance calculation

In general, the distance between two points measured in this metric along such a geodesic is:

$$dist(< x_1, y_1 >, < x_2, y_2 >) = cosh^{-1}(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2})$$

$$= 2sinh^{-1}\frac{1}{2}\sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{y_1 y_2}}$$

Distance calculation

$$= 2ln\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2\sqrt{y_1 y_2}}$$

where $cosh^{-1}$ and $sinh^{-1}$ are inverse hyperbolic functions

$$sinh^{-1}(x) = ln(x + \sqrt{x^2 + 1})$$

$$cosh^{-1}(x) = ln(x + \sqrt{x^2 - 1})$$

where $x \geq 1$

Distance calculation

Special cases:

$$dist(< x, y_1 >, < x, y_2 >) = \left| ln \frac{y_2}{y_1} \right| = |ln(y_2) - ln(y_1)|$$

$$dist(< x_1, y >, < x_2, y >) = cosh^{-1}(1 + \frac{(x_2 - x_1)^2}{2y^2})$$

$$= 2sinh^{-1}(\frac{|x_2 - x_1|}{2y})$$

Distance calculation

Another way to calculate the distance between two points that are on a (Euclidean) half circle is:

$$dist(AB) = \left| ln\left(\frac{|BA_\infty||AB_\infty|}{|AA_\infty||BB_\infty|}\right) \right|$$

where A_∞, B_∞ are the points where the halfcircles meet the boundary line and $|PQ|$ is the euclidean length of the line segment connecting the points P and Q in the model.

Distance in hyperbolic geometry

Definition

Let σ : [a, b] $\to \mathbb{H}$ be a path in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. Then the hyperbolic length of σ is obtained by integrating the function f(z) $=$1/ Im(z) along σ, i.e.

$$length_{\mathbb{H}}(\sigma) = \int_{\sigma} \frac{1}{Im(z)} = \int_{a}^{b} \frac{|\sigma'(t)|}{Im(\sigma(t))} dt$$

Example 1

Consider the path $\sigma(t) = a_1 + t(a_2 - a_1) + ib, 0 \leqslant t \leqslant 1$ between $a_1 + ib$ and $a_2 + ib$. Then σ'(t) = $a_2 - a_1$ and Im(σ(t)) = b. Hence

$$length_{\mathbb{H}}(\sigma) = \int_0^1 \frac{|a_2 - a_1|}{b} dt = \frac{|a_2 - a_1|}{b}$$

Consider the points -2 + i and 2 + i. By the example above, the length of the horizontal path between them is 4.

Example-continued

Now consider a different path from 2+i to 2+i. Consider the piecewise linear path that goes diagonally up from 2 + i to 2i and then diagonally down from 2i to 2 + i. A parametrisation of this path is given by

$$\sigma(t) = \begin{cases} (2t-2) + i(1+t), & 0 \leq \mathrm{t} \leq 1 \\ (2t-2) + i(3-t), & 1 \leq \mathrm{t} \leq 2 \end{cases}$$

Then

$$\sigma'(t) = \begin{cases} 2 + i, & 0 \leq \mathrm{t} \leq 1 \\ 2 - i, & 1 \leq \mathrm{t} \leq 2 \end{cases}$$

Example-continued

so that

$$\left|\sigma'(t)\right| = \begin{cases} |2+i| = \sqrt{5}, & 0 \leq \mathrm{t} \leq 1 \\ |2-i| = \sqrt{5}, & 1 \leq \mathrm{t} \leq 2 \end{cases}$$

and

$$Im(\sigma(t)) = \begin{cases} 1+t, & 0 \leq \mathrm{t} \leq 1 \\ 3-t, & 1 \leq \mathrm{t} \leq 2 \end{cases}$$

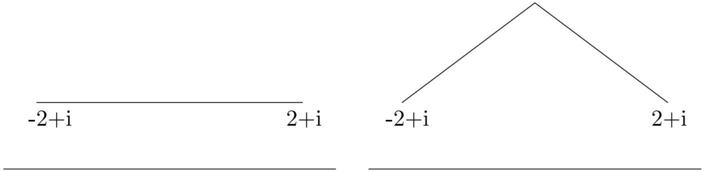
Hence,

$$length_{\mathbb{H}}(\sigma) = \int_0^1 \frac{\sqrt{5}}{1+t}dt + \int_1^2 \frac{\sqrt{5}}{3-t}dt$$

$$= [\sqrt{5}log(1+t)]_0^1 - [\sqrt{5}log(3-t)]_1^2 = 2\sqrt{5}log2 \approx 3.1$$

Conclusion of example

Note that the path from $2 + i$ to $2 + i$ in the third example has a shorter hyperbolic length than the path from $2 + i$ to $2 + i$ in the second example. This suggests that the geodesic (the paths of shortest length) in hyperbolic geometry are very different to the geodesics we are used to in Euclidean geometry.



Example 2

Consider the points i and a_i where $0 < a < 1$.
(i) Consider the path σ between i and a_i that consists of the arc of imaginary axis between them. Find a parametrisation of this path.
(ii) Show that $length_{\mathbb{H}}(\sigma) = log\frac{1}{a}$.

Answer:
(i): $\sigma : [a, 1] \to \mathbb{H}$ given by $\sigma(t) = it$. Then clearly $\sigma(a) = ia$ and $\sigma(1) = i$ (so that σ has the required end-points) and $\sigma(t)$ belongs to the imaginary axis.
(ii): Using (i), $|\sigma'(t) = 1|$ and $Im(\sigma(t)) = t.Hence,$

$$length_{\mathbb{H}}(\sigma) = \int_1^a \frac{1}{t} dt = [log(t)]_a^1 = -log(a) = log\frac{1}{a}$$

Distance in hyperbolic geometry

Definition

Let $A \subset \mathbb{R}$. A lower bound of A is any number $b \in \mathbb{R}$ such that b $\leq$ a for all $a \in A$. A lower bound l is called the infimum of A or greatest lower bound of A if it is greater than, or equal to, any other lower bound; that is, $b \leq l$ for all lower bounds b of A. We write inf A for the infimum of A, if it exists.

Examples

1. $\inf[1, 2] = 1$
2. $\inf(3, 4) = 3$
3. Infimum of a set A may not be an element of A.
Hence, infimum is different from minimum.
4. Infimum may not exist. The set $(-\infty, 0)$ does not have an infimum because it is unbounded from below.

Definition

Let z, z' $\in \mathbb{H}$. We define the hyperbolic distance $d_{\mathbb{H}}(z, z')$ between z and z' to be $d_{\mathbb{H}}(z, z') = \inf\{\text{length}_{\mathbb{H}}(\sigma) | \sigma$ is a piecewise continuously differentiable path with end-points z and z' $\}$.

Example

Show that $d_{\mathbb{H}}$ satisfies the triangle inequality:

$$d_{\mathbb{H}}(x,z) \leq d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z), \forall x,y,z \in \mathbb{H}$$

That is, the distance between two points is increased if one goes via a third point.

Idea

The distance between two points is the infimum of the (hyperbolic) lengths of (piecewise continuously differentiable) paths between them. Only a subset of these paths pass through a third point; hence the infimum of this subset is greater than the infimum over all paths.

Example - continued

Proof - continued

Let $x, y, z \in \mathbb{H}$. Let $\sigma_{x,y} : [a, b] \to \mathbb{H}$ be a path from x to y and let $\sigma_{y,z} : [b, c] \to \mathbb{H}$ be a path from y to z. Then the path $\sigma_{x,z} : [a, c] \to \mathbb{H}$ formed by defining

$$\sigma_{x,z}(t) = \begin{cases} \sigma_{x,y}, & t \in [a, b] \\ \sigma_{y,z}, & t \in [b, c] \end{cases}$$

is a path from x to z and has length equal to the sum of the lengths of $\sigma_{x,y}, \sigma_{y,z}$.

Example - continued

Proof - continued

Hence

$$d_{\mathbb{H}}(x,z) \leq length_{\mathbb{H}}(\sigma_{x,z}) = length_{\mathbb{H}}(\sigma_{x,y}) + length_{\mathbb{H}}(\sigma_{y,z})$$

Taking the infima over path from x to y and from y to z we see that

$$d_{\mathbb{H}}(x,z) \leq d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z)$$

Measurements in the Poincaré Disc Model

Möbius transformations of $\mathbb{D}$

Definition

A Möbius transformation is a function $m : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the form

$$m(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Let Möb^+ denote the set of all Möbius transformations.

Möbius transformations of $\mathbb{D}$

Theorem

Every element of $\text{Möb}(\mathbb{D})$ either has the form

$$p(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}},$$

or has the form,

$$p(z) = \frac{\alpha \overline{z} + \beta}{\overline{\beta}\,\overline{z} + \overline{\alpha}},$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$.

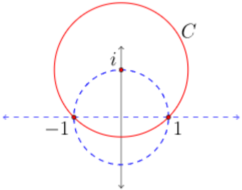
Möbius transformations of $\mathbb{D}$

Proof:

There are two forms of element of Möb($\mathbb{H}$),

- $m(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in R$ and $ad - bc = 1$
- $n(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ where a, b, c, d are purely imaginary and $ad - bc = 1$

Möbius transformation $p(z) = \frac{z-i}{-iz+1}$ takes $\overline{\mathbb{R}}$ to $\mathbb{S}^1$, and since $p(i) = 0$, $p(z)$ takes $\mathbb{H}$ to $\mathbb{D}$.



Möbius transformations of $\mathbb{D}$

For m, we calculate

$$p \circ m \circ p^{-1}(z) = \frac{(a+d+(b-c)i)z+b+c+(a-d)i}{(b+c-(a-d)i)z+a+d-(b-c)i} = \frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}},$$

where $\alpha = a+d+(b-c)i$ and $\beta = b+c+(a-d)i$.

For n, we calculate

$$p \circ n \circ p^{-1}(z) = \frac{(a-d-(b+c)i)z+b-c-(a+d)i}{(-b+c-(a+d)i)z-a+d-(b+c)i} = \frac{\delta\overline{z}+\gamma}{\overline{\gamma z}+\overline{\delta}},$$

where $\delta = a-d-(b+c)i$ and $\gamma = b-c-(a+d)i$.

Möbius transformations of $\mathbb{D}$

The Möbius transformation taking $\mathbb{D}$ to $\mathbb{D}$ are the elements of

$$\text{Möb}^+(\mathbb{D}) = \text{Möb}^+ \cap \text{Möb}(\mathbb{D}),$$

which are those elements of Möb($\mathbb{D}$) of the form

$$p(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}.$$

Möbius transformations of $\mathbb{D}$

Theorem

There is a unique Möbius transformation taking any three distinct points of $\overline{\mathbb{C}}$ to any three distinct points of $\overline{\mathbb{C}}$. That is:

$$m(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

where $m(z_1) = 0, m(z_2) = 1$, and $m(z_3) = \infty$.

Möbius transformations of $\mathbb{D}$

Example:

An explicit Möbius transformation taking $\mathbb{D}$ to $\mathbb{H}$

Möbius transformation m takes the triple $(i, -1, 1)$ of distinct points on $\mathbb{S}^1 = \partial\mathbb{D}$ to the triple $(0, 1, \infty)$ of distinct points on $\overline{\mathbb{R}} = \partial\mathbb{H}$. That is:

$$m(z) = \frac{z - i}{z - 1} \cdot \frac{-2}{-1 - i}.$$

Prove the imaginary part of $m(0)$ is positive:

$$m(0) = \frac{0 - i}{0 - 1} \cdot \frac{-2}{-1 - i} = \frac{2i}{1 + i} = 1 + i.$$

$$\mathrm{Im}(m(0)) = \mathrm{Im}(1 + i) = 1 > 0.$$

Möbius transformations of $\mathbb{D}$

Function:

$$\xi(z) = \frac{iz+1}{-z-i} = \frac{-2x}{x^2+(y+1)^2} + i\,\frac{1-x^2-y^2}{x^2+(y+1)^2}.$$

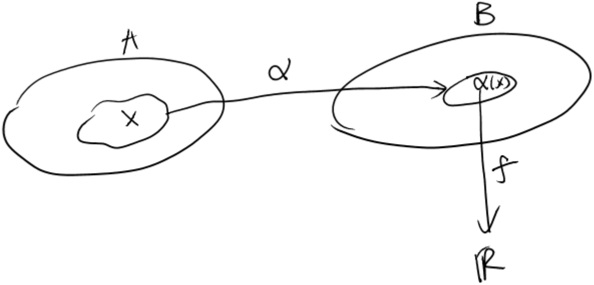
This is the function for transferring the hyperbolic geometry from $\mathbb{H}$ to $\mathbb{D}$.

Theorem

Suppose that $\mathbb{D}$ is an open subset of the complex plane $\mathbb{C}$ and that $\xi : \mathbb{D} \to \mathbb{H}$ is a diffeomorphism that is differentiable as a function of z. The pullback ds_X of the hyperbolic element of arc-length $\frac{1}{\mathrm{Im}(z)}|d(z)|$ on $\mathbb{H}$ is

$$\mathrm{ds}_X = \frac{1}{\mathrm{Im}(\xi(z))}|\xi'(z)||dz|.$$

Möbius transformations of $\mathbb{D}$



Möbius transformations of $\mathbb{D}$

Example:

Let $Y = \{z \in \mathbb{C} | \text{Re}(z) > 0\}$, and consider the diffeomorphism $\xi : Y \to \mathbb{H}$ given by $\xi(z) = i\ z$.

Since $\text{Im}(\xi(z)) = \text{Im}(iz) = \text{Re}(z)$ and $|\xi'(z)| = |i| = 1$, we see that

$$\text{ds}_X = \frac{1}{\text{Im}(\xi(z))} |\xi'(z)||dz| = \frac{1}{\text{Re}(z)} |dz|.$$

Hence, the pullback of $\frac{1}{\text{Im}(z)}|dz|$ is $\frac{1}{\text{Re}(z)}|dz|$.

Hyperbolic length and distance in $\mathbb{D}$

Theorem

The hyperbolic length of a piecewise differentiable path $f : [a, b] \to \mathbb{D}$ is given by the integral

$$\text{length}_{\mathbb{D}}(f) = \int_f \frac{2}{1 - |z|^2} |dz|.$$

The group of isometries of the resulting hyperbolic metric on $\mathbb{D}$ is $\text{Möb}(\mathbb{D})$.

Hyperbolic length and distance in $\mathbb{D}$

Proof:

An explicit element n of Möb taking $\mathbb{D}$ to $\mathbb{H}$:

$$n(z) = \frac{\frac{i}{\sqrt{2}} z + \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}} z - \frac{i}{\sqrt{2}}}.$$

Suppose $n \circ f : [a, b] \to \mathbb{H}$ is a piecewise differentiable path into $\mathbb{H}$.

We can calculate the hyperbolic length of $n \circ f$ by integrating $\frac{1}{\mathrm{Im}(z)} |dz|$ on $\mathbb{H}$ along $n \circ f$.

Hence, define the hyperbolic length of f in $\mathbb{D}$ by

$$\mathrm{length}_{\mathbb{D}}(f) = \mathrm{length}_{\mathbb{H}}(n \circ f)$$

Hyperbolic length and distance in $\mathbb{D}$

Then,

$$\begin{aligned}
\text{length}_{\mathbb{D}}(f) = \text{length}_{\mathbb{H}}(n \circ f) &= \int_{n \circ f} \frac{1}{\operatorname{Im}(z)} |dz| \\
&= \int_b^a \frac{1}{\operatorname{Im}((n \circ f)(t))} |(n \circ f)'(t)| dt \\
&= \int_b^a \frac{1}{\operatorname{Im}(n(f(t)))} |n'(f(t))||f'(t)| dt \\
&= \int_f \frac{1}{\operatorname{Im}(n(z))} |n'(z)||dz|.
\end{aligned}$$

Hyperbolic length and distance in $\mathbb{D}$

Then,

$$\mathrm{Im}(n(z)) = \mathrm{Im}\left(\frac{\frac{i}{\sqrt{2}}z + \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}z - \frac{i}{\sqrt{2}}}\right) = \frac{1-|z|^2}{|-z-i|^2}$$

and

$$|n'(z)| = \frac{2}{|z+i|^2}$$

and

$$\frac{1}{\mathrm{Im}(n(z))}|n'(z)| = \frac{2}{1-|z|^2}.$$

Hyperbolic length and distance in $\mathbb{D}$

Let p be any element of Möb taking $\mathbb{D}$ to $\mathbb{H}$.

Since $p \circ n^{-1}$ takes $\mathbb{H}$ to $\mathbb{H}$, we can set $q = p \circ n^{-1}$, so that q is an element of Möb($\mathbb{H}$).

Since $n \circ f$ is a piecewise differentiable path in $\mathbb{H}$, the invariance of hyperbolic length calculated with respect to $\frac{1}{\mathrm{Im}(z)}|dz|$ on $\mathbb{H}$ under Möb($\mathbb{H}$) implies that

$$\mathrm{length}_{\mathbb{H}}(n \circ f) = \mathrm{length}_{\mathbb{H}}(q \circ n \circ f) = \mathrm{length}_{\mathbb{H}}(p \circ f).$$

Hence, $\mathrm{length}_{\mathbb{D}}(f)$ is well-defined.

Hyperbolic length and distance in $\mathbb{D}$

Example:

Let $0 < \mathrm{r} < 1$ and consider the path $f : [0, r] \to \mathbb{D}$ given by $f(t) = t$. Then,

$$
\begin{aligned}
\text{length}_{\mathbb{D}}(f) &= \int_f \frac{2}{1 - |z|^2} |dz| \\
&= \int_0^r \frac{2}{1 - t^2} dt \\
&= \int_0^r [\frac{1}{1 + t} + \frac{1}{1 - t}] dt \\
&= \ln\, [\frac{1 + r}{1 - r}].
\end{aligned}
$$

That is the formula of the hyperbolic distance from 0 to a point r in $\mathbb{D}$.

Hyperbolic length and distance in $\mathbb{D}$

Definition

Given points x and y in $\mathbb{D}$, let $\Theta[x, y]$ be the set of all piecewise differentiable paths $f : [a, b] \to \mathbb{D}$ with $f(a) = x$ and $f(b) = y$.

In $\mathbb{D}$, the distance between x, y = the length of the shortest path between x, y, that is:

$$d_{\mathbb{D}}(x, y) = \inf\{ \text{length}_{\mathbb{D}}(f) : f \in \Theta[x, y] \}.$$

Fact:

Suppose $h : \mathbb{H} \to \mathbb{D}$ is a transformation of $\mathbb{H}$, h maps $\mathbb{H}$ bijectively to $\mathbb{D}$ and maps $\partial\mathbb{H}$ to $\partial\mathbb{D}$ bijectively.

Then $d_{\mathbb{D}}(h(z), h(w)) = d_{\mathbb{H}}(z, w)$.

Hyperbolic length and distance in $\mathbb{D}$

Satisfy the features of distance in hyperbolic geometry. That is:

For any x, y and $z \in \mathbb{D}$,

(1) $d_{\mathbb{D}}(x, y) > 0$.

(2) Shortest path between x, y is on the hyperbolic line connecting them.

(3) If x, y and z are three points on a hyperbolic line with y between the other two then $d_{\mathbb{D}}(x, y) + d_{\mathbb{D}}(y, z) = d_{\mathbb{D}}(x, z)$.

(4) Distance should be preserved by transformations in $\mathbb{D}$. the distance formula should satisfy $d_{\mathbb{D}}(x, y) = d_{\mathbb{D}}(m(x), m(y))$ for any transformation in $\mathbb{D}$.

Hyperbolic length and distance in $\mathbb{D}$

(1) Given two distnct points p and q inside the disc, the unique hyperbolic line connecting them intersects the boundary at two ideal points, a and b.

The hyperbolic distance between p and q is

$$d_{\mathbb{D}}(p,q) = \mathbf{ln}\frac{|aq||pb|}{|ap||qb|}.$$

(2) The hyperbolic distance from the centre to a point r in $\mathbb{D}$ is

$$d_{\mathbb{D}}(0,r) = \mathbf{ln}(\frac{1+|r|}{1-|r|}).$$

and

$$r = \tanh\,[\frac{1}{2}d_{\mathbb{D}}(0,r)]$$

Hyperbolic length and distance in $\mathbb{D}$

Proof of (2):

Parametrize the hyperbolic line segment between 0 and r by $f : [0, r] \to \mathbb{D}$ given by $f(t) = t$.

Since the image of f is the hyperbolic line segment in $\mathbb{D}$ joining 0 and r, then $d_{\mathbb{D}}(0, r) = \text{length}_{\mathbb{D}}(f)$.

$$d_{\mathbb{D}}(0, r) = \text{length}_{\mathbb{D}}(f) = \ln\,[\frac{1+r}{1-r}].$$

And,

$$\begin{aligned} d_{\mathbb{D}}(0, r) &= \ln\,[\frac{1+r}{1-r}] \\ \frac{1}{2} d_{\mathbb{D}}(0, r) &= \tanh^{-1}(r) \\ r &= \tanh[\frac{1}{2} d_{\mathbb{D}}(0, r)]. \end{aligned}$$

Hyperbolic length and distance in $\mathbb{D}$

An explicit example for φ which is invariant under Möb($\mathbb{D}$).

For any piecewise differentiable path $f : [a, b] \to \mathbb{D}$ and every element p of Möb$^+(\mathbb{D})$,

$$\begin{aligned}\int_a^b \frac{2}{1-|f(t)|^2}|f'(t)|dt &= \int_a^b \frac{2}{1-|(p\circ f)(t)|^2}|(p\circ f)'(t)|dt \\ &= \int_a^b \frac{2}{1-|p(f(t))|^2}|p'(f(t))|\;|f'(t)|dt\end{aligned}$$

Lemma
Let D be an open subset of $\mathbb{C}$, let $\mu : D \to \mathbb{R}$ be a continuous function, and suppose that $\int_f \mu(z)|dz| = 0$ for every pieceiwse differentiable path $f : [a, b] \to D$. Then, $\mu \equiv 0$.

Hyperbolic length and distance in $\mathbb{D}$

Then, for every element p of Möb$^+(\mathbb{D})$,

$$\frac{2}{1-|z|^2} = \frac{2|p'(z)|}{1-|p(z)|^2}.$$

For every pair x and y of points of $\mathbb{D}$ and every element p of Möb$^+(\mathbb{D})$,

$$(p(x)-p(y))^2 = p'(x)p'(y)(x-y)^2.$$

The form of Möbius transformation p is

$$p(z) = \frac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}},$$

where α, $\beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$.

Hyperbolic length and distance in $\mathbb{D}$

Then,

$$p(z) - p(t) = \frac{z - t}{(\overline{\beta} z + \overline{\alpha})(\overline{\beta} t + \overline{\alpha})}$$

and

$$p'(z) = \frac{1}{(\overline{\beta} z + \overline{\alpha})^2}.$$

Using those calculations, we have

$$\begin{aligned} \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)} &= |x - y|^2 \left(\frac{|p'(x)|}{1 - |p(x)|^2}\right)\left(\frac{|p'(y)|}{1 - |p(y)|^2}\right) \\ &= \frac{|p(x) - p(y)|^2}{(1 - |p(x)|^2)(1 - |p(y)|^2)}. \end{aligned}$$

Hyperbolic length and distance in $\mathbb{D}$

Hence,
The function $\varphi : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ defined by

$$\varphi(x, y) = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}$$

is invariant under the action of $\text{Möb}^+(\mathbb{D})$.

But why we need φ in $\mathbb{D}$?

Proposition

For each pair x and y of points of $\mathbb{D}$, we have that

$$\varphi(x, y) = \sinh^2(\frac{1}{2} d_{\mathbb{D}}(x, y)) = \frac{1}{2}(\cosh(d_{\mathbb{D}}(x, y)) - 1).$$

Hyperbolic length and distance in $\mathbb{D}$

Proof:

Let x and y be a pair of points in $\mathbb{D}$. Let $p(z) = \frac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}}$ be an element of $\text{Möb}^+(\mathbb{D})$ for which $p(x) = 0$.

Set $\beta = -\alpha x$, then

$$p(z) = \frac{\alpha(z-x)}{\overline{\alpha}(-\overline{x}z+1)},$$

where $|\alpha|^2(1-|x|^2) = 1$.

Hyperbolic length and distance in $\mathbb{D}$

Then, choose an α to make $p(y) = r$ which is real and positive. Hence,

$$\frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)} = \varphi(x,y)$$
$$= \varphi(p(x), p(y)) = \varphi(0, r) = \frac{r^2}{1-r^2}$$

Since $r = \tanh(\frac{1}{2} d_{\mathbb{D}}(0, r))$, then

$$\frac{r^2}{1-r^2} = \sinh^2 \left(\frac{1}{2} d_{\mathbb{D}}(x,y)\right) = \frac{1}{2} \left(\cosh(d_{\mathbb{D}}(x,y)) - 1\right).$$

Conclusion

	Upper half-plane	Poincaré disc
	$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$	$\mathbb{D} = \{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$
Boundary	$\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$	$\partial\mathbb{D} = \{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$
Length of a path σ	$\int_a^b \frac{1}{\mathrm{Im}\,\sigma(t)} \lvert\sigma'(t)\rvert\, dt$	$\int_a^b \frac{2}{1-\lvert\sigma(t)\rvert^2} \lvert\sigma'(t)\rvert\, dt$
Area of a subset A	$\int\int_A \frac{1}{(\mathrm{Im}\, z)^2}\, dz$	$\int\int_A \frac{4}{(1-\lvert z\rvert^2)^2}\, dz$
Orientation-preserving isometries	$\gamma(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$	$\gamma(z) = \frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}$, $\alpha, \beta \in \mathbb{C}$, $\lvert\alpha\rvert^2 - \lvert\beta\rvert^2 > 0$
Geodesics	vertical half-lines and semi-circles orthogonal to $\partial\mathbb{H}$	diameters of $\mathbb{D}$ and arcs of circles that meet $\partial\mathbb{D}$ orthogonally
Angles	Same as Euclidean angles	Same as Euclidean angles

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End

That's the end of our presentation.

Thank you!