MATH1520 University Mathematics for Applications **Fall 2021**

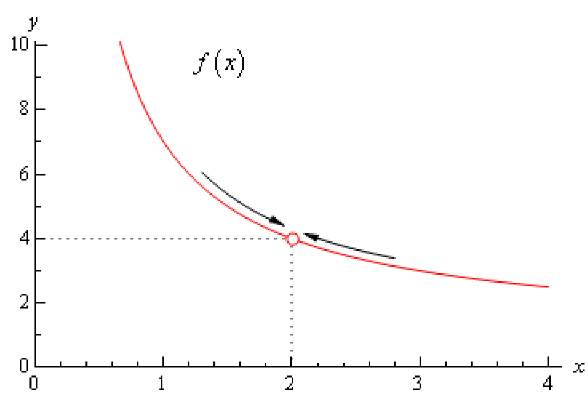
Chapter 2: Limits

Learning Objectives:
(1) Examine the limit concept and general properties of limits.
(2) Compute limits using a variety of techniques.
(3) Compute and use one-sided limits.
(4) Investigate limits involving infinity and "e".

2.1 Limit of a function at one point

(Heuristic) "Definition" 2.1.1. If $f(x)$ gets "closer and closer" to a number L as x gets "closer and closer" to c from *both sides*, then L is called the limit of $f(x)$ as x approaches c, denoted by

$$\lim_{x\to c} f(x) = L.$$



Remark. Limits are defined rigorously via "$\varepsilon - \delta$" language.

Example 2.1.1. Let $f(x) := x + 1$. Find $\lim_{x\to 1} f(x)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x\to 1} f(x) = 2$.

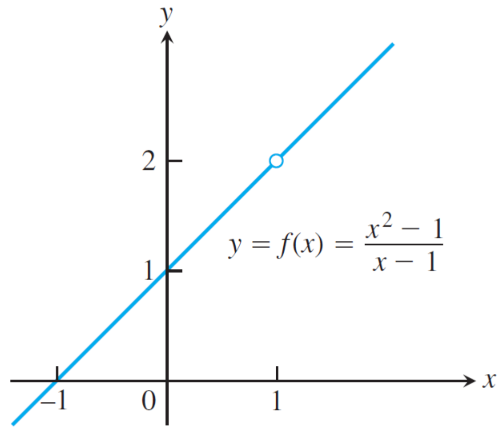
Remark. 1. The table only gives you an intuitive idea, this is not a rigorous proof.
2. Don't think that the limit is always obtained by substituting $x = 1$ into $f(x)$. The limit only depends on the behavior of $f(x)$ near $x = 1$, but not at $x = 1$.

Example 2.1.2. $f(x) = \begin{cases} x+1 & \text{if } x \neq 1, \\ \text{undefined} & \text{if } x = 1. \end{cases}$

x	0.9	0.99	0.999	**1**	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	**undefined**	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x\to 1} f(x) = 2$.

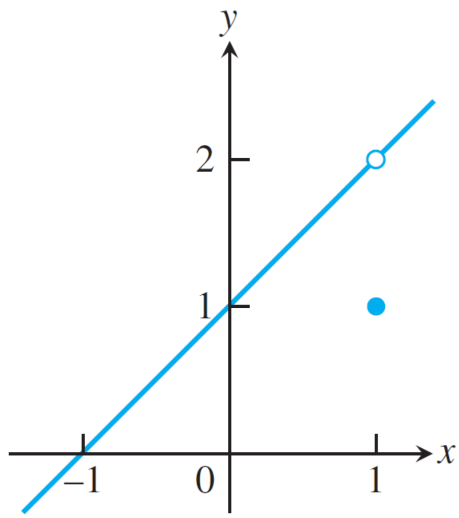
Disregard the value of f at 1, the limit of $f(x)$ when x tends to 1 is always 2.



Example 2.1.3. $f(x) = \begin{cases} x+1 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$

x	0.9	0.99	0.999	**1**	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	**1**	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x\to 1} f(x) = 2$.



Proposition 1.

1. If $f(x) = k$ is a constant function, then
$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k.$$
For instance, $\lim_{x\to 1} 9 = 9$.
2. If $f(x) = x$, then
$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c.$$
For instance, $\lim_{x\to 3} x = 3$.

Proposition 2. **(Algebraic properties of limits, $+, -, \times, \div$)**

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist (important!), then

1. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. $\lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
Especially, $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ for any constant k
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$.
5. $\lim_{x\to c}(f(x))^p = \left[\lim_{x\to c} f(x)\right]^p$ if $\left[\lim_{x\to c} f(x)\right]^p$ exists

Example 2.1.4. Compute the following limits:

1. $\lim_{x \to 1} (x^3 + 2x - 5)$
2. $\lim_{x \to 2} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$
3. $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution.

1. $\lim_{x \to 1} (x^3 + 2x - 5) = \lim_{x \to 1} x^3 + \lim_{x \to 1} 2x - \lim_{x \to 1} 5 = 1^3 + 2 \cdot 1 - 5 = -2.$
2. $\lim_{x \to 2} \dfrac{x^4 + x^2 - 1}{x^2 + 5} = \dfrac{\lim_{x \to 2} (x^4 + x^2 - 1)}{\lim_{x \to 2} (x^2 + 5)} = \dfrac{\lim_{x \to 2} x^4 + \lim_{x \to 2} x^2 - \lim_{x \to 2} 1}{\lim_{x \to 2} x^2 + \lim_{x \to 2} 5} = \dfrac{19}{9}.$
3. $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2} (4x^2 - 3)} = \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3} = \sqrt{16 - 3} = \sqrt{13}.$

■

Remark. Generalizing the arguments for the first example above: the limit of any polynomial function $P(x)$,

$$\lim_{x \to c} P(x) = P(c).$$

Exercise 2.1.1. Compute the following limits:

$$\lim_{x \to 1} \frac{1}{x - 1}; \qquad \lim_{x \to 1} \left(x^2 - \frac{3x}{x + 5} \right)$$

Example 2.1.5. (Cancelling a common factor)
Find the limit:

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 3x + 2}.$$

Solution. We can't directly use property of division of limit because the denominator $\lim_{x \to 1} (x^2 - 3x + 2) = 1^2 - 3 \times 1 + 2 = 0$.

$$\begin{aligned}
\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 3x + 2} &= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{(x - 1)(x - 2)} \\
&= \lim_{x \to 1} \frac{\cancel{(x - 1)}(x + 1)}{\cancel{(x - 1)}(x - 2)} \\
&= \lim_{x \to 1} \frac{x + 1}{x - 2} \\
&= \frac{1 + 1}{1 - 2} = -2.
\end{aligned}$$

■

Example 2.1.6. Compute

$$\lim_{x\to 1}\frac{x^3-5x+4}{x^2+2x-3}.$$

Solution. Write $p(x)=x^3-5x+4$ and $q(x)=x^2+2x-3$. Because $p(1)=q(1)=0$, $x-1$ is a factor of $p(x)$ and $q(x)$. We obtain

$$p(x)=(x-1)(x^2+x-4) \text{ and } q(x)=(x-1)(x+3).$$

Then

$$\begin{aligned}\lim_{x\to 1}\frac{x^3-5x+4}{x^2+2x-3}&=\lim_{x\to 1}\frac{(x-1)(x^2+x-4)}{(x-1)(x+3)}\\&=\lim_{x\to 1}\frac{x^2+x-4}{x+3}\\&=\frac{1^2+1-4}{1+3}=-\frac{1}{2}.\end{aligned}$$

■

Example 2.1.7. (Rationalization)
Let $f:[0,\infty)\backslash\{1\}\to\mathbf{R}$ defined by $f(x)=\dfrac{\sqrt{x}-1}{x-1}$. Find $\lim\limits_{x\to 1} f(x)$.

Solution. For $x\neq 1$.

$$\frac{\sqrt{x}-1}{x-1}=\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\frac{x-1}{(x-1)(\sqrt{x}+1)}=\frac{1}{\sqrt{x}+1}.$$

Hence

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}=\frac{1}{2}.$$

■

Example 2.1.8. (Rationalization and Cancellation)
Find

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x^2-1}.$$

Solution.

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{x^2-1}&=\lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x+1)(x-1)(\sqrt{x}+1)}\\&=\lim_{x\to 1}\frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}(\sqrt{x}+1)}\\&=\lim_{x\to 1}\frac{1}{(x+1)(\sqrt{x}+1)}=\frac{1}{4}.\end{aligned}$$

■

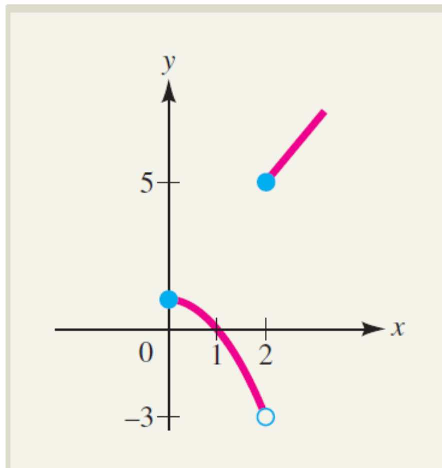
Challenge Question: Let $f : \mathbf{R}\backslash\{1\} \to \mathbf{R}$ defined by $f(x) = \dfrac{\sqrt[3]{x} - 1}{x - 1}$.
Find $\lim_{x \to 1} f(x)$.
Hint: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

Proposition 3 (Composite functions/change of variables)**.** If $\lim_{x \to c} g(x) = k$ exists and $\lim_{u \to k} f(u)$ exists, then $\lim_{x \to c} f \circ g(x) = \lim_{u \to k} f(u)$.

Example 2.1.9. Redo the last three examples using change of variables.

2.2 One-sided Limits

The following shows the graph of a piecewise function $f(x)$:



As x approaches 2 from the right, $f(x)$ approaches 5 and we write

$$\lim_{x \to 2^+} f(x) = 5.$$

On the other hand, as x approaches 2 from the left, $f(x)$ approaches -3 and we write

$$\lim_{x \to 2^-} f(x) = -3.$$

Limits of these forms are called one-sided limits. The limit is a right-hand limit if the approach is from the right. From the left, it is a left-hand limit.

Definition 2.2.1. If $f(x)$ approaches L as x tends towards c from the left ($x < c$), we write $\lim_{x \to c^-} f(x) = L$. It is called the **left-hand limit** of $f(x)$ at c.
If $f(x)$ approaches L as x tends towards c from the right ($x > c$), we write $\lim_{x \to c^+} f(x) = L$. It is called the **right-hand limit** of $f(x)$ at c.

Example 2.2.1. Recall

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

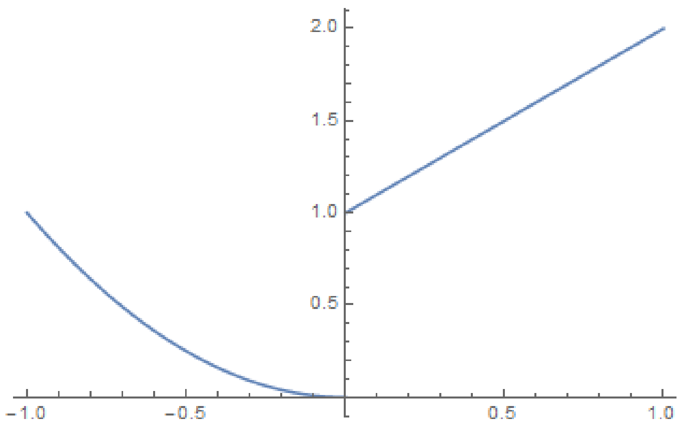
$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0.$$

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0.$$

For this case $\lim_{x\to 0^+} |x| = \lim_{x\to 0^-} |x|$. Then $\lim_{x\to 0} |x| = 0$.

Example 2.2.2. Define $f : \mathbf{R} \to \mathbf{R}$,

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^{-2}	10^{-4}	10^{-6}	1	1.001	1.01	1.1

We have

$$\lim_{x\to 0^+} f(x) = 1.$$

and

$$\lim_{x\to 0^-} f(x) = 0.$$

Remark.

1. The left hand limit or the right hand limit may not be the same.

2. Does $\lim_{x\to 0} f(x)$ exist? No!

Proposition 4.

$\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$.

(i.e., both left hand limit and right hand limit exist and is equal to L)

Example 2.2.3. Suppose the function

$$f(x) = \begin{cases} x^2 + 1, & x \geq 1, \\ a, & x < 1. \end{cases}$$

has a limit as x approaches 1. Find the value of a and $\lim_{x \to 1} f(x)$.

Solution. Since $\lim_{x \to 1} f(x)$ exists, we have

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1} f(x).$$

And

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 1) = 2, \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (a) = a.$$

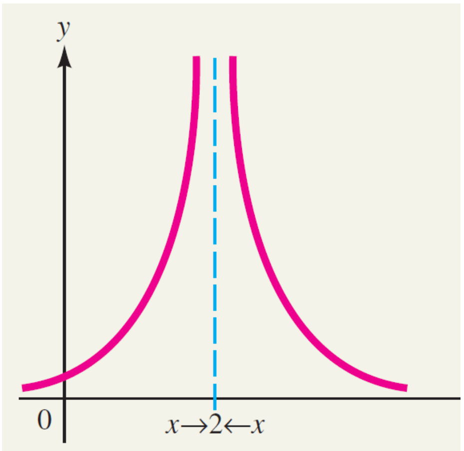
So, $a = 2$, and $\lim_{x \to 1} f(x) = 2$. ■

2.3 Infinite "Limits"

Consider the following limit

$$\lim_{x \to 2} \frac{1}{(x-2)^2}.$$

As x approaches 2, the denominator of the function $f(x) = \dfrac{1}{(x-2)^2}$ approaches 0 and hence the value of $f(x)$ becomes very large.



The function $f(x)$ increases without bound as $x \to 2$ both from left and from right. In this case, the limit *DNE (does not exist)* at $x = 2$, but we express the asymptotic behaviour of f near 2 *symbolically* as

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = +\infty.$$

Remark. $+\infty$ is just a symbol, not a real number.

Example 2.3.1.

$$\lim_{x \to 0} \frac{-1}{x^2} = -\infty.$$

Definition 2.3.1. We say that $\lim_{x \to c} f(x)$ is an infinite limit if $f(x)$ increases or decreases without bound as $x \to c$.
If $f(x)$ increases without bound as $x \to c$, we write

$$\lim_{x \to c} f(x) = +\infty.$$

If $f(x)$ decreases without bound as $x \to c$, then

$$\lim_{x \to c} f(x) = -\infty.$$

Example 2.3.2. Evaluate

$$\lim_{x \to 2^+} \frac{x-3}{x^2-4} \text{ and } \lim_{x \to 2^-} \frac{x-3}{x^2-4}.$$

Solution.

$$\lim_{x \to 2^+} \frac{x-3}{x^2-4} = \lim_{x \to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$$

since as $x \to 2^+$, we have $x^2 - 4 = (x-2)(x+2) \to 0^+$ and $x - 3 \to -1^+$.

$$\lim_{x \to 2^-} \frac{x-3}{x^2-4} = \lim_{x \to 2^-} \frac{x-3}{(x-2)(x+2)} = +\infty$$

since as $x \to 2^-$, we have $x^2 - 4 = (x-2)(x+2) \to 0^-$ and $x - 3 \to -1^-$.

■

Exercise 2.3.1. Find

$$\lim_{x \to \pi/2} \tan x; \qquad \lim_{x \to \pi/2^-} \tan x; \qquad \lim_{x \to \pi/2^+} \tan x; \qquad \lim_{x \to 0^+} \ln x.$$

Remark. Caveat! When applying the rules in Proposition 2, roughly speaking:

- "$a \pm \infty = \pm\infty$" when a is finite;
- "$\infty + \infty = \infty$"; "$-\infty - \infty = -\infty$";
- "$\infty \cdot \infty = \infty$"; "$-\infty \cdot \infty = -\infty$"; "$-\infty \cdot (-\infty) = \infty$";
- "$a \cdot \infty = \text{sign}(a)\,\infty$" when $a \neq 0$;
- "$\dfrac{a}{\pm\infty} = 0$" when a is finite;
- "$\dfrac{a}{0^{\pm}} = \pm\text{sign}(a)\,\infty$" when $a \neq 0$;
- but "$\infty - \infty$", "$0 \cdot \infty$", "$\dfrac{\infty}{\infty}$", "$\dfrac{0}{0}$" can be quite arbitrary, and must be determined case by case! We shall introduce tools to compute limits of these forms later.

2.4 Limits at Infinity

Definition 2.4.1. If the values of the function $f(x)$ approach the number L as x gets bigger and bigger (i.e. as x goes to $+\infty$). Then L is called the limit of $f(x)$ as x tends to $+\infty$. Denoted by

$$\lim_{x \to +\infty} f(x) = L.$$

Similarly we can define

$$\lim_{x \to -\infty} f(x) = M.$$

Remark: The value L and M may not be the same.

Example 2.4.1. Let $f(x) = \dfrac{1}{x}$.

-1000	-100	-10	-1	1	10	100	1000
-0.001	-0.01	-0.1	-1	1	0.1	0.01	0.001

$$\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to +\infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0.$$

Proposition 5. If A and $k > 0$ are constants. Then

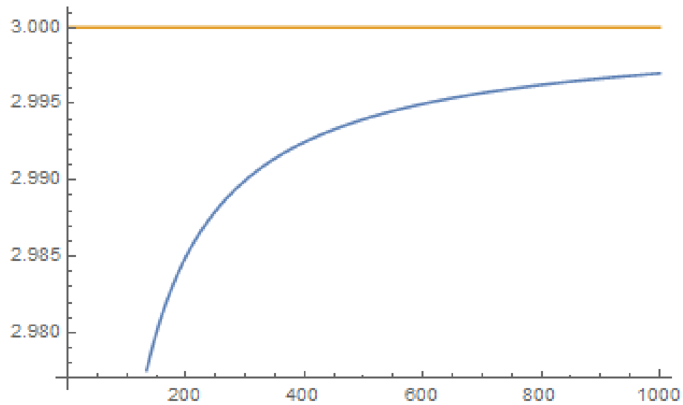
$$\boxed{\lim_{x \to +\infty} \frac{A}{x^k} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{A}{x^k} = 0.}$$

To determine the limit of a rational function as $x \to \pm\infty$, we can divide the numerator and denominator by the highest power of x in the denominator.

Example 2.4.2. Find $\lim\limits_{x\to+\infty} \dfrac{3x^2}{x^2+x+1}$

Solution.

$$\begin{aligned} &\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1} \quad \text{(Divide both the top and bottom by } x^2\text{)} \\ =&\lim_{x\to+\infty} \frac{3}{1+\frac{1}{x}+\frac{1}{x^2}} \\ =&\frac{3}{1+0+0} = 3. \end{aligned}$$



■

Question: Can we write

$$\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1} = \frac{\lim\limits_{x\to+\infty}(3x^2)}{\lim\limits_{x\to+\infty}(x^2+x+1)}?$$

Hint: Recall the Caveat from the end of last section.

Example 2.4.3. Find $\lim\limits_{x\to+\infty} \dfrac{x-1}{2x^2+3x+1}$

Solution.

$$\begin{aligned} &\lim_{x\to+\infty} \frac{x-1}{2x^2+3x+1} \quad \text{(Divide both the top and bottom by } x^2\text{)} \\ =&\lim_{x\to+\infty} \frac{\frac{1}{x}-\frac{1}{x^2}}{2+3\frac{1}{x}+\frac{1}{x^2}} \\ =&\frac{0}{2+0+0} = 0. \end{aligned}$$

■

Example 2.4.4. Find $\lim_{x\to+\infty} \dfrac{x^3-1}{2x^2+3x+1}$.

Solution.

$$\begin{aligned} &\lim_{x\to+\infty} \frac{x^3-1}{2x^2+3x+1} \\ =&\lim_{x\to+\infty} \frac{x-\frac{1}{x^2}}{2+\frac{3}{x}+\frac{1}{x^2}} \\ =&+\infty. \end{aligned}$$

■

Proposition 6. Suppose

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, a_n \neq 0$$

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0, b_m \neq 0$$

Then

$$\lim_{x\to+\infty} \frac{p(x)}{q(x)} = \begin{cases} \frac{a_n}{b_m} & \text{if } n = m, \\ 0 & \text{if } n < m, \\ +\infty & \text{if } a_n b_m > 0,\, n > m, \\ -\infty & \text{if } a_n b_m < 0,\, n > m. \end{cases}$$

Remark. One way to see this: The leading term in a polynomial dominates the lower order terms as $x \to \pm\infty$. (Higher powers of x "grows faster" than lower powers of x as $x \to \infty$. Log functions grows slower than any polynomial function because (as we'll see later) $\lim_{x\to\infty} \dfrac{\ln x}{x^a} = 0$ for any $a > 0$.

Example 2.4.5. Find $\lim_{x\to\infty} \dfrac{3x^3-2x^2+1}{-x^3+7}$.

Solution. By the proposition, the answer is $\frac{3}{-1} = -3$. ■

Similar technique can be used for functions with radical (i.e., something like $\sqrt{x}$).

Example 2.4.6. Find $\lim_{x\to+\infty} \dfrac{3x-1}{\sqrt{3x^2+1}}$.

Solution. The term with highest degree of the denominator is x^2. But we need to take square root. So we divide the nominator and the denominator by $\sqrt{x^2} = x$. We have

$$\lim_{x\to+\infty} \frac{3x-1}{\sqrt{3x^2+1}} = \lim_{x\to+\infty} \frac{\frac{1}{x}(3x-1)}{\frac{1}{x}\sqrt{3x^2+1}}$$

$$= \lim_{x\to+\infty} \frac{3-\frac{1}{x}}{\sqrt{3+\frac{1}{x^2}}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

■

Example 2.4.7. (Rationalization)
Evaluate

$$\lim_{x\to+\infty} (\sqrt{x+1} - \sqrt{x}).$$

Solution. (Recall the *Caveat* from last section!)

$$\begin{aligned}\lim_{x\to+\infty} (\sqrt{x+1} - \sqrt{x}) &= \lim_{x\to+\infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}} \\ &= \lim_{x\to+\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} \\ &= 0.\end{aligned}$$

■

Exercise 2.4.1.

1. $\lim_{x\to-\infty} \frac{x^3+1}{-2x^3+x} = -\frac{1}{2}$.
2. $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$ (Caution: $x < 0 \Rightarrow \frac{1}{x} = -\sqrt{\frac{1}{x^2}}$).
3. $\lim_{x\to+\infty} (\sqrt{x^2+x} - \sqrt{x^2-2}) = \frac{1}{2}$.

Example 2.4.8. $\lim_{x\to+\infty} \sin x = ?$

2.5 Limits involving "e"

Definition 2.5.1.

$$e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x.$$

e is the base for natural log, $\log_e x = \ln x$.

$$e = 2.71828\dots$$

x	-1000	-100	-10	10	100	1000
$\left(1+\frac{1}{x}\right)^x$	2.71964	2.73200	2.86797	2.59374	2.70481	2.71692

Remark. 1. Note that

$$e := \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x \neq \left(\lim_{x\to+\infty} 1+\frac{1}{x}\right)^x = 1!$$

2. Motivation for defining e this wa will be clear later when we learn about differentiation.

Example 2.5.1. Evaluate

$$\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x.$$

Solution.

$$\begin{aligned}\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x &= \lim_{x\to+\infty}\left[\left(1+\frac{1}{(-x)}\right)^{(-x)}\right]^{-1} \qquad (\text{ set } -x=y)\\ &= \left[\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y\right]^{-1}\\ &= e^{-1}\end{aligned}$$

■

Exercise 2.5.1. Evaluate $\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^{2x} = e^4$.