THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics
Solution to Coursework 9

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Problem 1. Evaluate the indefinite integral

$$\int \frac{\cos(5x)}{1+\sin^2(5x)}dx.$$

Solution. Recall that $\int \cos(x)dx = \sin(x) + C$, where C is a constant. Then we have

$$\int \frac{\cos(5x)}{1+\sin^2(5x)}dx = \int \frac{1}{5}\frac{d\sin(5x)}{1+\sin^2(5x)}.$$

Since $\int \frac{1}{1+x^2}dx = \text{arctan }(x) + C$, let $t = \sin(5x)$, we have

$$\int \frac{1}{5}\frac{d\sin(5x)}{1+\sin^2(5x)} = \int \frac{1}{5}\frac{dt}{1+t^2} = \frac{1}{5}\text{arctan }(t) + C.$$

We then know that

$$\int \frac{\cos(5x)}{1+\sin^2(5x)}dx = \frac{1}{5}\text{arctan }\big(\sin(5x)\big) + C.$$

Problem 2. Evaluate the indefinite integral

$$\int \sin^3(5x)\cos^{10}(5x)dx.$$

Solution. Recall that through the Pythagorean Identity $\sin^2(x) = 1-\cos^2(x)$. Then we know that $\sin^3(5x) = \sin(5x)\big(\sin^2(5x)\big) = \sin(5x)\big(1-\cos^2(5x)\big)$. Substituting this into the integral we see

$$\int \sin^3(5x)\cos^{10}(5x)dx = \int \sin(5x)\big(1-\cos^2(5x)\big)\cos^{10}(5x)dx.$$

Distributing just the cosines, this becomes

$$\int \sin(5x)\big(\cos^{10}(5x) - \cos^{12}(5x)\big)dx.$$

Now use the substitution $t = \cos(5x) \Rightarrow dt = d\cos(5x) = -5\sin(5x)dx$, i.e., $\sin(5x)dx = -\frac{1}{5}dt$, then the integral becomes

$$\int \big(\cos^{10}(5x) - \cos^{12}(5x)\big)\sin(5x)dx = \int (t^{10} - t^{12})(-\frac{1}{5})dt = \frac{1}{5}(\frac{t^{13}}{13} - \frac{t^{11}}{11}) + C.$$

Reordering and back-substituting with $t = \cos(5x)$, we get

$$\int \sin^3(5x)\cos^{10}(5x)dx = \frac{1}{5}(\frac{\cos^{13}(5x)}{13} - \frac{\cos^{11}(5x)}{11}) + C.$$

Problem 3. Evaluate the integral

$$\int \sin^4(x)dx.$$

Solution. This integral is mostly about clever rewriting of your functions. As a rule of thumb, if the power is even, we use the double angle formula. The double angle formula says

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta)).$$

If we split up our integral like this

$$\int \sin^2(x) \cdot \sin^2(x)dx$$

We can use the double angle formula twice:

$$\int \frac{1}{2}(1 - \cos(2x)) \cdot \frac{1}{2}(1 - \cos(2x))dx$$

Both parts are the same, so we can just put it as a square:

$$\int \left(\frac{1}{2}(1 - \cos(2x))\right)^2 dx$$

Expanding, we get:

$$\int \frac{1}{4}\left(1 - 2\cos(2x) + \cos^2(2x)\right) dx$$

We can then use the other double angle formula

$$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$$

to rewrite the last term as follows:

$$\begin{aligned}
&\frac{1}{4}\int 1 - 2\cos(2x) + \frac{1}{2}(1 + \cos(4x))dx \\
=&\frac{1}{4}\left(\int 1dx - \int 2\cos(2x)dx + \frac{1}{2}\int 1 + \cos(4x)dx\right) \\
=&\frac{1}{4}\left(x - \int 2\cos(2x)dx + \frac{1}{2}\left(x + \int \cos(4x)dx\right)\right)
\end{aligned}$$

We will call the left integral in the parenthesis Integral 1, and the right on Integral 2.

Integral 1: $\int 2\cos(2x)dx$

Looking at the integral, we have the derivative of the inside, 2 outside of the function, and this should immediately ring a bell that you should use u-substitution. If we let $u = 2x$, the derivative becomes 2, so we divide through by 2 to integrate with respect to u

$$\int \frac{\cancel{2}\cos(u)}{\cancel{2}}du = \int \cos(u)du,$$

Integral 2: $\int \cos(4x)dx$

$$\int \cos(u)du = \sin(u) = \sin(2x).$$

It's not as obvious here, but we can also use u-substitution here. We can let $u = 4x$, and the derivative will be 4

$$\frac{1}{4}\int \cos(u)dx = \frac{1}{4}\sin(u) = \frac{1}{4}\sin(4x).$$

Completing the original integral Now that we know Integral 1 and Integral 2, we can plug them back into our original expression to get the final answer

$$\begin{aligned}
&\frac{1}{4}\left(x-\sin(2x)+\frac{1}{2}\left(x+\frac{1}{4}\sin(4x)\right)\right)+C\\
=&\frac{1}{4}\left(x-\sin(2x)+\frac{1}{2}x+\frac{1}{8}\sin(4x)\right)+C\\
=&\frac{1}{4}x-\frac{1}{4}\sin(2x)+\frac{1}{8}x+\frac{1}{32}\sin(4x)+C\\
=&\frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C.
\end{aligned}$$

Problem 4. Let a be a nonzero real number. Evaluate the integral

$$\int \frac{-7x}{x^4-a^4}dx.$$

Solution. As a is a nonzero real number, we divide the numerator and denominator by a^4 as

$$\int \frac{\frac{-7x}{a^4}}{(\frac{x}{a})^4-1}dx.$$

Similar to Problem 1, let $t=\frac{x}{a}$, we get

$$\int \frac{\frac{-7x}{a^4}}{(\frac{x}{a})^4-1}dx=\frac{-7}{a^2}\int \frac{\frac{x}{a}}{(\frac{x}{a})^4-1}d(\frac{x}{a})=\frac{-7}{a^2}\int \frac{t}{t^4-1}dt.$$

Note that $\int tdt=\int \frac{1}{2}dt^2$, we rewrite this integral as

$$\frac{-7}{a^2}\int \frac{t}{t^4-1}dt=\frac{-7}{2a^2}\int \frac{1}{t^4-1}dt^2.$$

Let $k=t^2$, we have

$$\frac{-7}{2a^2}\int \frac{1}{k^2-1}dk.$$

With the partial decomposition, we know that

$$\frac{-7}{2a^2}\int \frac{1}{k^2-1}dk=\frac{-7}{2a^2}\int \frac{1}{2(k-1)}-\frac{1}{2(k+1)}dx=\frac{-7}{4a^2}(\ln|k-1|-\ln|k+1|)+C.$$

Reordering and back-substituting with $k=t^2$, we get

$$\frac{-7}{4a^2}(\ln|t^2-1|-\ln|t^2+1|)+C.$$

Reordering and back-substituting with $t=\frac{x}{a}$, we get the final answer

$$\frac{-7}{4a^2}(\ln|(\frac{x}{a})^2-1|-\ln|(\frac{x}{a})^2+1|)+C.$$

Problem 5. Evaluate the integral when $x>0$

$$\int \ln\left(x^2+11x+24\right)dx.$$

Solution. Integrate by parts using the formula $\int udv=uv-\int vdu$, where $u=\ln\left(x^2+11x+24\right)$ and $dv=1$. Then we have

$$\ln\left(x^2+11x+24\right)x-\int x\;d\big(\ln\left(x^2+11x+24\right)\big)=\ln\left(x^2+11x+24\right)x-\int x\frac{2x+11}{(x+3)(x+8)}dx.$$

Combine x and $\frac{2x+11}{(x+3)(x+8)}$, the integral becomes

$$\ln\left(x^2+11x+24\right)x-\int\frac{2x^2+11x}{(x+3)(x+8)}dx$$

Note that $(x+3)(x+8)=x^2+11x+24$, therefore the integral becomes

$$\ln\left(x^2+11x+24\right)x-\int\frac{2x^2+11x}{x^2+11x+24}dx$$

Let $\frac{2x^2+11x}{x^2+11x+24}$ be more simpler as

$$\frac{2x^2+11x}{x^2+11x+24}=\frac{2(x^2+11x+24)-11x-48}{x^2+11x+24}=\frac{2(x^2+11x+24)}{x^2+11x+24}+\frac{-11x-48}{x^2+11x+24},$$

Apply the constant rule, we have

$$\ln\left(x^2+11x+24\right)x-\int\left(2+\frac{-11x-48}{x^2+11x+24}\right)dx=\ln\left(x^2+11x+24\right)x-\left(2x+C+\int\frac{-11x-48}{x^2+11x+24}dx\right)$$

Write the fraction using partial fraction decomposition, the integral becomes

$$\ln\left(x^2+11x+24\right)x-\left(2x+C+\int-\frac{3}{x+3}-\frac{8}{x+8}dx\right)$$

Split the single integral into multiple integrals

$$\ln\left(x^2+11x+24\right)x-\left(2x+C+\int-\frac{3}{x+3}dx+\int-\frac{8}{x+8}dx\right)$$

Let $u_1=x+3$. Then $du_1=dx$. Rewrite using u_1 and du_1.

$$\ln\left(x^2+11x+24\right)x-\left(2x+C-3\int\frac{1}{u_1}du_1+\int-\frac{8}{x+8}dx\right)$$

The integral of $\frac{1}{u_1}$ with respect to u_1 is $\ln(|u_1|)$.

$$\ln\left(x^2+11x+24\right)x-\left(2x+C-3\left(\ln(|u_1|)+C\right)+\int-\frac{8}{x+8}dx\right)$$

Let $u_2=x+8$. Then $du_2=dx$. Rewrite using u_2 and du_2.

$$\ln\left(x^2+11x+24\right)x-\left(2x+C-3\left(\ln(|u_1|)+C\right)-8\int\frac{1}{u_2}du_2\right)$$

The integral of $\frac{1}{u_2}$ with respect to u_2 is $\ln(|u_2|)$.

$$\ln\left(x^2+11x+24\right)x-\left(2x+C-3\left(\ln(|u_1|)+C\right)-8\left(\ln(|u_2|)+C\right)\right)$$

Simplify.

$$\ln\left(x^2+11x+24\right)x-2x+3\ln(|u_1|)+8\ln(|u_2|)+C$$

Substitute back in for each integration substitution variable.

Replace all occurrences of u_1 with $x+3$.

$$\ln\left(x^2+11x+24\right)x-2x+3\ln(|x+3|)+8\ln(|u_2|)+C$$

Replace all occurrences of u_2 with $x+8$, then we get the final solution

$$\ln\left(x^2+11x+24\right)x-2x+3\ln(|x+3|)+8\ln(|x+8|)+C.$$

Problem 6. Suppose that $f(1)=-10, f(4)=8, f'(1)=9, f'(4)=-7$, and f'' is continuous. Find the value of $\int_1^4 xf''(x)dx$

Solution. Integrate by parts using the formula $\int udv = uv - \int vdu$, where $u = x$ and $dv = f''(x)$. Then we have

$$\begin{aligned}
&\int_1^4 xf''(x)dx \\
=&xf'(x)|_1^4 - \int_1^4 f'(x)dx \\
=&xf'(x)|_1^4 - f(x)|_1^4 \\
=&\big(4 \cdot f'(4) - 1 \cdot f'(1)\big) - \big(f(4) - f(1)\big) \\
=&4 \cdot (-7) - 9 - 8 + (-10) \\
=& - 28 - 9 - 8 - 10 \\
=& - 55
\end{aligned}$$

Problem 7. Evaluate the integral

$$\int_{\pi/6}^{\pi/3} 8\csc^3(x)dx.$$

Solution. Integrate by parts using the formula $\int udv = uv - \int vdu$, where $u = \csc(x)$ and $dv = \csc^2(x)$, $du = -\csc(x)\cot(x)dx$, $v = -\cot(x)$, then

$$-\csc(x)\cot(x) - \int \cot^2(x)\csc(x)dx.$$

Note that $cot^2(x) = csc^2(x) - 1$, we have

$$-\csc(x)\cot(x) - \int \big(\csc^2(x) - 1\big)\csc(x)dx = -\csc(x)\cot(x) - \int \big(\csc^3(x) - \csc(x)\big)\, dx.$$

Then

$$\int \csc^3(x)dx = -\csc(x)\cot(x) - \int \csc^3(x)dx + \int \csc(x)dx.$$

Let the right side $-\int \csc^3(x)$ to the left,

$$2\int \csc^3(x)dx = -\csc(x)\cot(x) + \int \csc(x)dx,$$

i.e.,

$$\int \csc^3(x)dx = \frac{-\csc(x)\cot(x) + \int \csc(x)dx}{2}.$$

Note that $\int \csc(x)dx = \ln(\csc(x) - \cot(x)) + C1$, where $C1$ is a constant, then

$$\int \csc^3(x)dx = \frac{-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x))}{2} + C.$$

Therefore

$$\begin{aligned}
&\int_{\pi/6}^{\pi/3} 8\csc^3(x)dx \\
=&8\frac{-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x))}{2}\Big|_{\pi/6}^{\pi/3} \\
=&4(-\csc(\pi/3)\cot(\pi/3) + \ln(\csc(\pi/3) - \cot(\pi/3))) - 4(-\csc(\pi/6)\cot(\pi/6) + \ln(\csc(\pi/6) - \cot(\pi/6))) \\
=& -\frac{8}{3} - 4\ln\sqrt{3} + 8\sqrt{3} - 4\ln(2 - \sqrt{3}) \\
=& -\frac{8}{3} + 8\sqrt{3} - 4\ln(2\sqrt{3} - 3)
\end{aligned}$$

Problem 8. A rumor is spread in a school. For $0 < a < 1$ and $b > 0$, the time t at which a fraction p of the school population has heard the rumor is given by

$$t(p) = \int_a^p \frac{b}{x(1-x)}dx.$$

(a) Evaluate the integral to find an explicit formula for $t(p)$. Write your answer so it has only one ln **term.** $\int_a^p \frac{b}{x(1-x)} dx =$______________**.**
(b) At time $t = 0$, five percent of the school population ($p = 0.05$) has heard the rumor. What is a ? $a =$______________**.**
(c) At time $t = 1$, fifty-one percent of the school population ($p = 0.51$) has heard the rumor. What is b ? $b =$______________**.**
(d) At what time has ninety-four percent of the population ($p = 0.94$) heard the rumor? $t =$______________**.**

Solution. We integrate to find

$$\int \frac{b}{x(1-x)} dx = b \int \left(\frac{1}{x} + \frac{1}{1-x} \right) dx = b(\ln|x| - \ln|1-x|) + C = b \ln \left| \frac{x}{1-x} \right| + C_1,$$

then

$$t(p) = \int_a^p \frac{b}{x(1-x)} dx = b \ln \left(\frac{p}{1-p} \right) - b \ln \left(\frac{a}{1-a} \right) = b \ln \left(\frac{p(1-a)}{a(1-p)} \right).$$

(b) We know that $t(0.05) = 0$, so

$$0 = b \ln \left(\frac{0.05(1-a)}{0.95a} \right).$$

But $b > 0$ and $\ln x = 0$ means $x = 1$, so

$$\frac{0.05(1-a)}{0.95a} = 1, \quad \text{or} \quad 0.05(1-a) = 0.95a.$$

Solving $a = 005$.
(c) We know that $t(0.51) = 1$ so

$$1 = b \ln \left(\frac{0.51 \cdot 0.95}{0.49 \cdot 0.05} \right) = b \ln \frac{0.4845}{0.0245},$$

so $b = \frac{1}{\ln \frac{0.4845}{0.0245}}$.
(d) We have

$$t(0.94) = \int_{0.05}^{0.94} \frac{b}{x(1-x)} dx = \frac{1}{\ln \frac{0.4845}{0.0245}} \ln \left(\frac{0.94(1-0.05)}{0.05(1-0.94)} \right) \approx 1.9086$$

Problem 9. The German mathematician Karl Weierstrass (1815-1897) noticed that the substitution $t = \tan(x/2)$ will convert any rational function of $\sin(x)$ and $\cos(x)$ into an ordinary rational function of t. If $t = \tan(x/2), -\pi < x < \pi$, then it can be shown that

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}}, \quad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$$
$$\cos(x) = \frac{1-t^2}{1+t^2}, \quad \sin(x) = \frac{2t}{1+t^2},$$

and

$$dx = \frac{2}{1+t^2} dt.$$

Use the substitution given above to transform the following integral into a rational function of t and then evaluate the integral:

$$\int_{\pi/3}^{\pi/2} \frac{10}{1 + \sin(x) - \cos(x)} dx$$

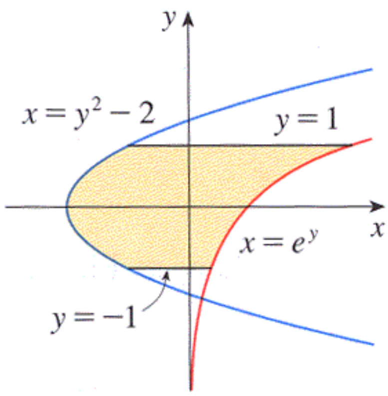
Solution. Let $\tan(x/2) = t$, then as the problem described, we know that when $x = \pi/3$, $t = \tan(\pi/6) = 1/\sqrt{3}$ and when $x = \pi/2$, $t = \tan(\pi/4) = 1$. Therefore

$$\begin{aligned}
\int_{\pi/3}^{\pi/2} \frac{10}{1+\sin x - \cos x} dx &= \int_{1/\sqrt{3}}^{1} \frac{10}{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\
&= \int_{1/\sqrt{3}}^{1} \frac{10(1+t^2)}{1+t^2+2t-1+t^2} \cdot \frac{2}{1+t^2} dt \\
&= \int_{1/\sqrt{3}}^{1} \frac{10}{t^2+t} dt \\
&= \int_{1/\sqrt{3}}^{1} \frac{10}{t(t+1)} dt \\
&= \int_{1/\sqrt{3}}^{1} 10\left(\frac{1}{t} - \frac{1}{t+1}\right) dt \\
&= 10[\ln|t| - \ln|t+1|]_{1/\sqrt{3}}^{1} \\
&= 10\left[\ln\left|\frac{t}{t+1}\right|\right]_{1/\sqrt{3}}^{1} \\
&= 10\Big(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1/\sqrt{3}}{1\sqrt{3}+1}\right)\Big) \\
&= 10\Big(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{\sqrt{3}+1}\right)\Big) \\
&= 10\ln\left(\frac{\sqrt{3}+1}{2}\right).
\end{aligned}$$

Problem 10. Evaluate $\int_{\pi/6}^{\pi} |\cos(x)|dx$. Solution.

$$\begin{aligned}
\int_{\pi/6}^{\pi} |\cos x| dx &= \int_{\pi/6}^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x) dx \\
&= \sin x|_{\pi/6}^{\pi/2} - \sin x|_{\pi/2}^{\pi} \\
&= 1 - 0.5 - (0-1) \\
&= 1.5
\end{aligned}$$

Problem 11. Find the area of the shaded region below. Solution.



$$\begin{aligned}
\text{Area} &= \int_{-1}^{1}\int_{y^2-2}^{e^y} dxdy \\
&= \int_{-1}^{1} x|_{y^2-2}^{e^y}\ dy \\
&= \int_{-1}^{1} e^y - y^2 + 2\ dy \\
&= e^y - \frac{1}{3}y^3 + 2y\ |_{-1}^{1} \\
&= (e^1 - \frac{1}{3}\cdot 1 + 2\cdot 1) - (e^{-1} - \frac{1}{3}\cdot(-1) + 2\cdot(-1)) \\
&= e - \frac{1}{e} + \frac{10}{3}.
\end{aligned}$$

Problem 12. Given

$$f(x) = \int_0^x \frac{t^2-36}{1+\cos^2(t)}dt.$$

At what value of x does the local max of $f(x)$ occur?

Solution. First, note that we don't need to do any computation to compute the first derivative, which we will use to check for local maxima and minima. By applying the Fundamental Theorem of Calculus, we see that:

$$f'(x) = \frac{x^2-36}{1+\cos^2(x)}.$$

Now, we can use this derivative to find the critical points of the function. We set this to zero and solve for x to get:

$$\begin{aligned}
\frac{x^2-36}{1+\cos^2(x)} &= 0 \\
x^2-36 &= 0 \\
(x+6)(x-6) &= 0 \\
x = 6 \text{ or } x &= -6
\end{aligned}$$

Checking on either side of these two points shows that -6 is the local maximum for which we are looking.

Problem 13. Part 1: A derivative computation using the chain rule Suppose $F(x)$ is any function that is differentiable for all real numbers x. Evaluate the following derivative.

$\frac{d}{dx}\left(F\left(x^4\right)\right) =$ ________

Enter the derivative of $F(x)$ as F′(x) using prime notation. Your answer should be in terms of F' and other functions of the variable x.

Part 2: A derivative computation using the FTC

Suppose $F(x) = \int_{11}^{x} e^{-t^2}dt$. Use the Fundamental Theorem of Calculus to evaluate the derivative.

$$F'(x) = \frac{d}{dx}\left(\int_{11}^{x} e^{-t^2}dt\right) =$$

Part 3: A composition of two functions.

Suppose $F(x) = \int_{11}^{x} e^{-t^2}dt$. Find a formula for the function $F\left(x^4\right)$ expressed using an integral.

$$F\left(x^4\right) =$$

Part 4: A derivative computation using the FTC and the chain rule

$$\frac{d}{dx}\left(F\left(x^4\right)\right) = \frac{d}{dx}\left(\int_{11}^{x^4} e^{-t^2}dt\right) =$$

Solution. Part 1. $\frac{d}{dx}\left(F\left(x^4\right)\right) = 4x^3F'(x^4)$

Part 2. Using the Fundamental Theorem of Calculus which states if

$$F(x) = \int_a^x f(t)\mathrm{d}t,$$

then

$$F'(x) = f(x).$$

Hence

$$F'(x) = \frac{d}{dx}\left(\int_{11}^{x} e^{-t^2}dt\right) = e^{-x^2}.$$

Part 3.

$$F\left(x^4\right) = \int_{11}^{x^4} e^{-t^2}dt.$$

Part 4.

$$\frac{d}{dx}(F(x^4)) = \frac{d}{dx}\Big(\int_{11}^{x^4} e^{-t^2}dt\Big) = 4x^3e^{-x^8}.$$

Problem 14. Find the following limit using l'Hopital's Rule:

$$\lim_{x\to 0^+} \frac{\int_0^x \sqrt{t}\cos t dt}{x^2}$$

Solution. l'Hopital's Rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Then we know that

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x).$$

Then we get

$$\Big(\int_0^x \sqrt{t}\cos t dt\Big)' = (\sqrt{x}\cos x)\cdot 1 - 0\cdot 1\cdot 0 = \sqrt{x}\cos x,$$

then

$$\lim_{x\to 0^+}\frac{\int_0^x \sqrt{t}\cos t dt}{x^2} = \lim_{x\to 0^+}\frac{\sqrt{x}\cos x}{2x} = \lim_{x\to 0^+}\frac{\cos x}{2\sqrt{x}} = \frac{1}{0^+} = \infty.$$

Problem 15. Suppose that $F(x) = \int_1^x f(t)dt$, where

$$f(t) = \int_1^{t^2} \frac{\sqrt{7+u^6}}{u}du.$$

Find $F''(2)$.

$$F''(2) = \_\_\_\_\_\_\_$$

Solution. Using the Fundamental Theorem of Calculus which states if

$$F(x) = \int_a^x f(t)\mathrm{d}t,$$

then

$$F'(x) = f(x).$$

Thus

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_1^x f(t)\mathrm{d}t = f(x),$$

$$F''(x) = f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_1^{x^2} \frac{\sqrt{7+u^6}}{u}\,\mathrm{d}u = \frac{2x\sqrt{7+x^{12}}}{x^2}$$

$$\Rightarrow F''(2) = \frac{2 \cdot 2\sqrt{7+2^{12}}}{2^2} = \sqrt{4103}.$$

Problem 16. Let $F(x) = \int_0^x \frac{3-t}{t^2+55} dt$ for $-\infty < x < +\infty$.
(a) Find the value of x where F obtains its maximum value.

$$x = \_\_\_\_\_\_\_$$

(b) Find the intervals over which F is only increasing or decreasing. Use interval notation using U for union and enter "none" if no interval.
Intervals where F is increasing:
Intervals where F is decreasing:
(c) Find open intervals over which F is only concave up or concave down. Use interval notation using U for union and enter "none" if no interval.
Intervals where F is concave up:
Intervals where F is concave down:

Solution.
(a)

$$F'(x) = \frac{d}{dx} \int_0^x \frac{3-t}{t^2+55} dt = \frac{3-x}{x^2+55} = 0,$$

when $x = 3$ which is the only critical point. From sign analysis of F' we see this is a maximum.
(b) F is increasing on $(-\infty, 3]$ and decreasing on $[3, +\infty)$.
(c)

$$F''(x) = \frac{d}{dx}\left[\frac{3-x}{x^2+55}\right] = \frac{x^2-6x-55}{(x^2+55)^2} = \frac{(x-11)(x+5)}{(x^2+55)^2} = 0,$$

when $x = -5, 11$. Sign analysis of F'' shows that F is concave up on $(-\infty, -5)$ and $(11, +\infty)$ and concave down on $(-5, 11)$.

Problem 17. Evaluate the integral

$$\int_{-1}^{2} (4x - 5|x|) dx$$

Solution. We split the interval $[-1, 2]$ to $[-1, 0]$ and $[0, 2]$. Then we get

$$\begin{aligned}
\int_{-1}^{2} (4x-5|x|)dx &= \int_{-1}^{0} (4x-5|x|)dx + \int_{0}^{2} (4x-5|x|)dx \\
&= \int_{-1}^{0} (4x+5x)dx + \int_{0}^{2} (4x-5x)dx \\
&= \int_{-1}^{0} 9x dx + \int_{0}^{2} (-x)dx \\
&= \frac{9}{2}x^2|_{-1}^0 - \frac{1}{2}x^2|_0^2 \\
&= 0 - \frac{9}{2} \cdot (-1)^2 - \frac{1}{2} \cdot (2)^2 \\
&= -\frac{9}{2} - \frac{4}{2} \\
&= -\frac{13}{2}
\end{aligned}$$

Problem 18. Consider the function $f(x) = \frac{x^2}{4} + 7$. In this problem you will calculate $\int_0^2 \left(\frac{x^2}{4} + 7\right) dx$ by using the definition

$$\int_a^b f(x)dx = \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i) \Delta x\right].$$

The summation inside the brackets is R_n, which is the Riemann sum where the sample points are chosen to be the right-hand endpoints of each sub-interval.
Calculate R_n for $f(x) = \frac{x^2}{4} + 7$ on the interval $[0, 2]$ and write your answer as a function of n without any summation signs.

$$R_n = \_\_\_\_\_\_\_$$

$$\lim_{n\to\infty} R_n = \_\_\_\_\_\_\_$$

Solution. Note

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

So we calculate Δx :

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$

And although the problem does not specify to use right endpoints, if you check the hint you will find out that's what they intend, so we let $x_i = 0 + \Delta x \cdot i = \frac{2i}{n}$. And then we plug it in and simplify:

$$\begin{aligned}
R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x &= \sum_{i=1}^{n} \left(\frac{2}{n}\right) f\left(\frac{2i}{n}\right) \\
&= \frac{2}{n} \cdot \Big(\sum_{i=1}^{n} \left[\frac{\left(\frac{2i}{n}\right)^2}{4}\right] + 7\Big) \\
&= \frac{2}{n} \cdot \sum_{i=1}^{n} \left(1 \cdot \frac{i^2}{n^2} + 7\right) \\
&= \frac{2}{n} \cdot \left[\left(\sum_{i=1}^{n} \frac{i^2}{n^2}\right) + \left(\sum_{i=1}^{n} 7\right)\right] \\
&= \frac{2}{n} \cdot \left[\left(\frac{1}{n^2} \sum_{i=1}^{n} i^2\right) + 7n\right]
\end{aligned}$$

Using the standard summation formulation

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \qquad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6},$$

then we get

$$\begin{aligned}
&\frac{2}{n}\left[\frac{1}{n^2}\frac{n(n+1)(2n+1)}{6} + 7n\right] \\
&= \frac{2}{n}\left[\frac{1}{n^2}\frac{2n^3+3n^2+n}{6} + 7n\right] \\
&= 2\left[\frac{1}{n^3}\frac{2n^3+3n^2+n}{6} + 7\right] \\
&= 2\left[\frac{2n^3+3n^2+n+42n^3}{6n^3}\right] \\
&= \frac{44n^3+3n^2+n}{3n^3} \\
&= \frac{44+\frac{3}{n}+\frac{1}{n^2}}{3}.
\end{aligned}$$

Then we know that

$$R_n = \frac{44+\frac{3}{n}+\frac{1}{n^2}}{3}.$$

Take the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{44+\frac{3}{n}+\frac{1}{n^2}}{3} = \frac{44+0+0}{3} = \frac{44}{3}.$$

Problem 19. Consider the integral $\int_2^6 \frac{x}{1+x^5} dx$. Which of the following expressions represents the integral as a limit of Riemann sums?

A. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}$

B. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)}$

C. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \frac{2+\frac{6i}{n}}{1+\left(2+\frac{6i}{n}\right)}$

D. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}$

E. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2+\frac{6i}{n}}{1+\left(2+\frac{6i}{n}\right)^5}$

F. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \frac{2+\frac{6i}{n}}{1+\left(2+\frac{6i}{n}\right)^5}$

Solution. Riemann Sum: Given a function $f(x)$ defined on $[a,b]$ and a partition $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ Then, a Riemann sum is a sum of the form,

$$\sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k$$

Where $x_k^* \in [x_{k-1}, x_k]$ and $\Delta x_k = x_k - x_{k-1}$. When $x_k^* = x_k$, we call the sum the right Riemann sum.

Let $f(x) = \frac{x}{1+x^5}$ and $a = 2, b = 6$ We will use the right Riemann Sum with a fixed length sub-interval. Then $\Delta x = x_k - x_{k-1} = \frac{b-a}{n} = \frac{4}{n}$ and $x_k = 2 + k * \Delta x = \frac{2n+4k}{n}$. So,

$$f\left(x_k\right) = \frac{\frac{2n+4k}{n}}{1+\left(\frac{2n+445}{n}\right)^5} = \frac{n^4(2n+4k)}{n^5+(2n+4k)^5}.$$

So, the right Riemann sum is given by

$$\sum_{k=1}^{n} f\left(x_k\right) \Delta x_k = \sum_{k=1}^{n} \frac{n^4(2n+4k)}{n^5+(2n+4k)^5} \frac{4}{n} = \sum_{k=1}^{n} \frac{4n^3(2n+4k)}{n^5+(2n+4k)^5}.$$

Therefore,

$$\int_2^6 \frac{x}{1+x^5} dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_k\right) \Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{4n^3(2n+4k)}{n^5+(2n+4k)^5} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{4}{n} \frac{2+\frac{4k}{n}}{1+\left(2+\frac{4k}{n}\right)^5}.$$