THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2022-2023 Term 1
Suggested Solutions of WeBWork Coursework 6

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(1) Find the critical point and determine if the function is increasing or decreasing on the given intervals.

$$y = -x^2 + 6x + 3$$

The critical point c is? And determine the monotonicity of y on

$$(-\infty, c),\ (c, \infty)$$

Solution:
The derivative of $f(x) = -x^2 + 6x + 3$ is $f'(x) = -2x + 6$.
This means $y' = 0$ when $x = 3$.
Therefore $c = 3$ is the critical point.
Note that

$$y' = -2x + 6 < 0 \text{ when } x > 3$$
$$y' = -2x + 6 > 0 \text{ when } x < 3$$

So y is increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$.

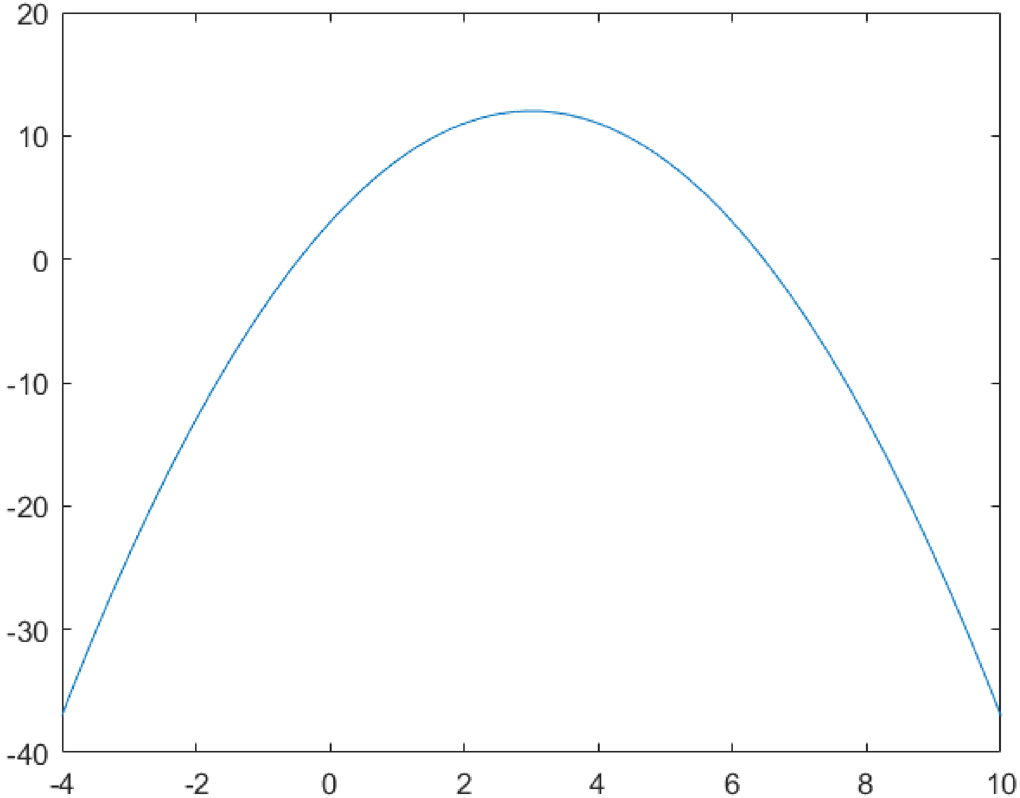


FIGURE 1. The graph of $f(x) = -x^2 + 6x + 3$

(2) Find the critical point and the interval on which the given function is increasing or decreasing, and apply the First Derivative Test to the critical point. Let

$$f(x) = 20\ln(10x) - 4x, x > 0$$

(a) Critical Point =?

(b) Is f a maximum or minimum at the critical point ?

(c) The **open** interval on the left of the critical point is ?
On this interval, f is ? (increasing/decreasing) while f' is ? (positive or negative)

(d) The **open** interval on the right of the critical point is ?
On this interval, f is ? (increasing/decreasing) while f' is ? (positive or negative)

Solution:

(a) First we need to calculate f',

$$f' = 20 \cdot 10 \frac{1}{10x} - 4$$

So $f'(x) = 0$ when $x = 5$. So the critical point is $x = 5$.

(b) Let us compute the second derivative

$$f''(x) = -\frac{20}{x^2}$$

Since $f''(x) < 0$ for all $x > 0$, it follows that $f'(x)$ is always decreasing. So $f'(x)$ is positive on $(0,5)$ and negative on $(5,\infty)$. That means $f(x)$ is increasing on $(0,5)$ and decreasing on $(5,\infty)$. Hence $f(x)$ is maximum at the critical point $x = 5$.

(c) As shown in (b), the interval on the left of the critical point is $(0,5)$, $f(x)$ is increasing and $f'(x)$ is positive on this interval.

(d) As shown in (b), the interval on the right of the critical point is $(5,\infty)$, $f(x)$ is decreasing and $f'(x)$ is negative on this interval.

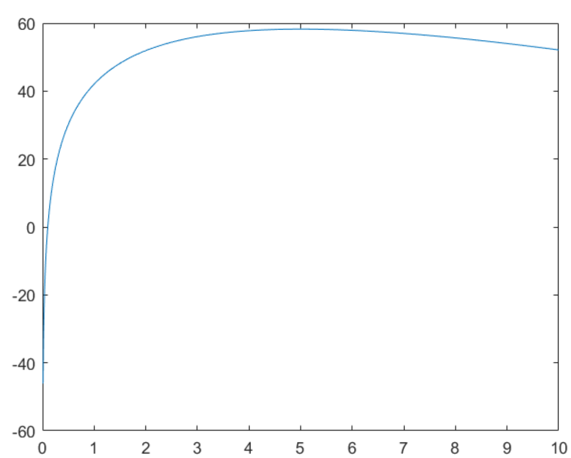


FIGURE 2. The graph of $f(x) = 20\ln(10x) - 4x$

(3) The function

$$f(x) = -6x^3 - 54x^2 + 126x - 7$$

is increasing on the interval [___ , ___].
It is decreasing on the interval ($-\infty$, ___] and the interval [___, ∞).

Solution:
We can compute the derivate of $f(x)$:

$$f'(x) = -18x^2 - 108x + 126$$

Let us find the critical point:

$$-18x^2 - 108x + 126 = -18(x-1)(x+7) = 0 \Longrightarrow x = -7 \text{ or } x = 1$$

The graph of $f'(x)$: So we have

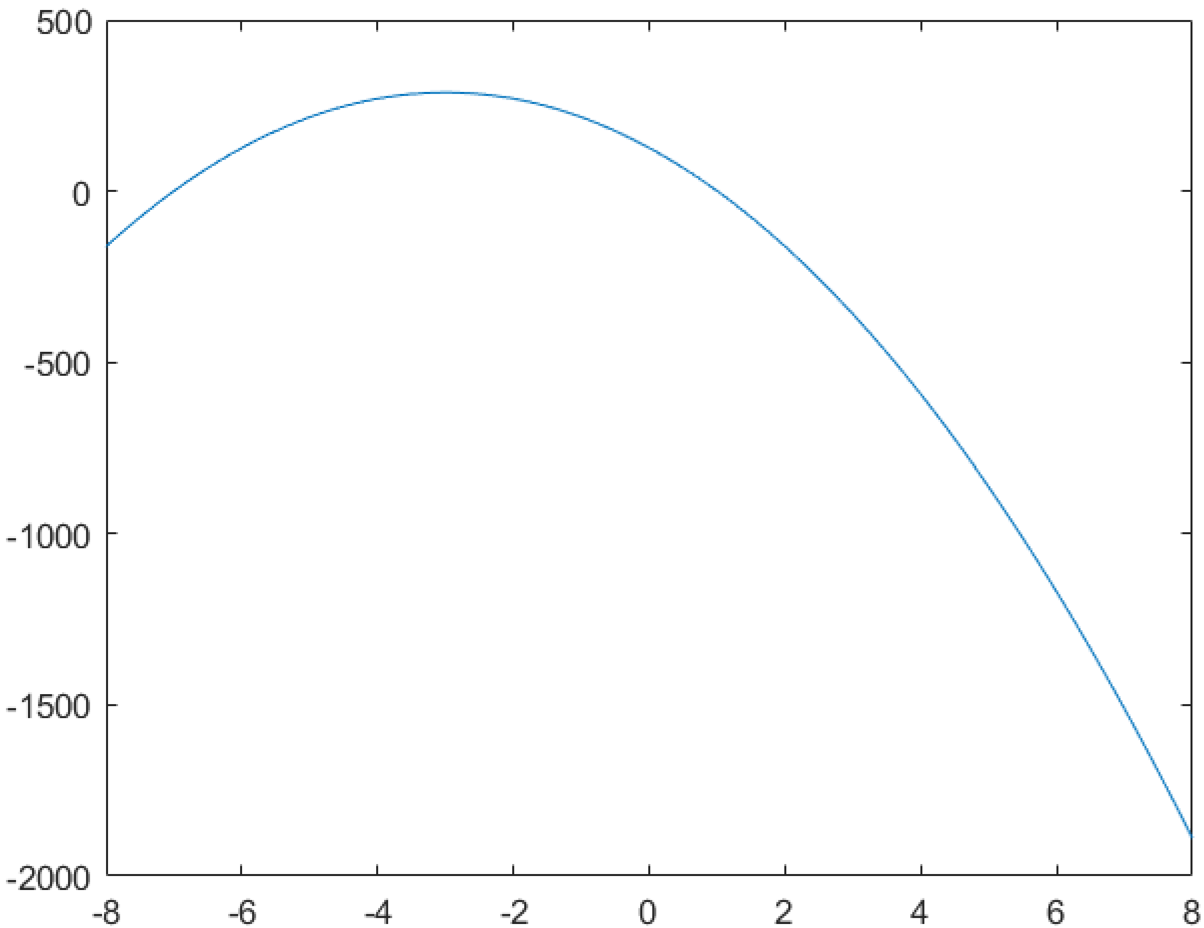


FIGURE 3. $f'(x)$

x	$x < -7$	$-7 < x < 1$	$x > 1$
$f'(x)$	Negative	Positive	Negative
$f(x)$	Decreasing	Increasing	Decreasing

So $f(x)$ is increasing on the interval $[-7, 1]$ and decreasing on the intervals $(-\infty, -7]$ and $[1, \infty)$.

(4)

$$f(x) = \frac{x}{x^2 + 10x + 24}$$

a) Give the domain of f (in interval notation) ________

b) Determine the intervals on which f is increasing and decreasing.

f is increasing on: ________

f is decreasing on: ________

Solution:

(a) Since the denominator cannot be 0, the domain of $f(x)$ is where $x^2 + 10x + 24 = (x+6)(x+4) \neq 0$. So the domain of $f(x)$ is

$$(-\infty, -6) \cup (-6, -4) \cup (-4, \infty)$$

(b) Note that

$$f'(x) = \frac{-x^2 + 24}{(x^2 + 10x + 24)^2}$$

Solving $f'(x) = 0$, we have $x = -2\sqrt{6}$ or $x = 2\sqrt{6}$.

We should then have

x	$x < -2\sqrt{6}$	$-2\sqrt{6} < x < 2\sqrt{6}$	$x > 2\sqrt{6}$
$f'(x)$	Negative	Positive	Negative
$f(x)$	Decreasing	Increasing	Decreasing

However, $f(x)$ is not defined on $x = -4$ and $x = -6$, so we have to remove these two points from our result. Comparing the values, we have $-6 < -2\sqrt{6} < -4 < 2\sqrt{6}$. Therefore the correct chart is

x	$x < -6$	$-6 < x < -2\sqrt{6}$	$-2\sqrt{6} < x < -4$	$-4 < x < 2\sqrt{6}$	$x > 2\sqrt{6}$
$f'(x)$	Negative	Negative	Positive	Positive	Negative
$f(x)$	Decreasing	Decreasing	Increasing	Increasing	Decreasing

Hence $f(x)$ is increasing on

$$[-2\sqrt{6}, -4), (-4, 2\sqrt{6}]$$

and decreasing on

$$(-\infty, -6), (-6, -2\sqrt{6}], [2\sqrt{6}, \infty)$$

(5) Let $f(x) = 8\sqrt{x} - 4x$ for $x > 0$. Find the open intervals on which f is increasing (decreasing).

1. f is increasing on the intervals __________
2. f is decreasing on the intervals __________

Notes: In the first two, your answer should either be a single interval, such as (0,1), a comma separated list of intervals, such as (-inf, 2), (3,4), or the word “none”.

Solution:

Computing the derivative, we have

$$f'(x) = \frac{4}{\sqrt{x}} - 4$$

Note that $f'(x) = 0$ if $x = 1$. So we have

x	$0 < x < 1$	$x > 1$
$f'(x)$	Positive	Negative
$f(x)$	Increasing	Decreasing

So $f(x)$ is increasing on $(0,1)$ and decreasing on $(1,\infty)$.

(6) Consider the functions $f(x) = e^{x-1} - 1$ and $g(x) = x - 1$. These are continuous and differentiable for $x > 0$. In this problem we use the Racetrack Principle to show that one of these functions is greater than the other, except at one point where they are equal.

(a) Find a point c such that $f(c) = g(c)$. $c =$ ________

(b) Find the equation of the tangent line to $f(x) = e^{x-1} - 1$ at $x = c$ for the value of c that you found in (a).

$y =$ ____________________

(c) Based on your work in (a) and (b), what can you say about the derivatives of f and g?

$f'(x)$?[</=/>] $g'(x)$ for $0 < x < c$, and

$f'(x)$?[</=/>] $g'(x)$ for $c < x < \infty$.

(d) Therefore, the Racetrack Principle gives

$f(x)$?[<=/=/>=] $g(x)$ for $x \leq c$, and

$f(x)$?[<=/=/>=] $g(x)$ for $x \geq c$.

Solution:

(a) Note that at $c = 1$ we have $f(c) = g(c) = 0$.

(b) At $x = 1$ $f'(x) = e^{x-1}$, so that $f'(1) = 1$, and the equation of the tangent line is $y = x - 1$.

(c) As $f'(x) = e^{x-1}$, $g'(x) = 1$. Then we note that for $0 < x < 1$ we have $f'(x) < g'(x) = 1$ and for $x > 1$, that $f'(x) > g'(x) = 1$.

(d) Therefore, by the Racetrack Principle, we know that $f(x) \geq g(x)$ at every x.

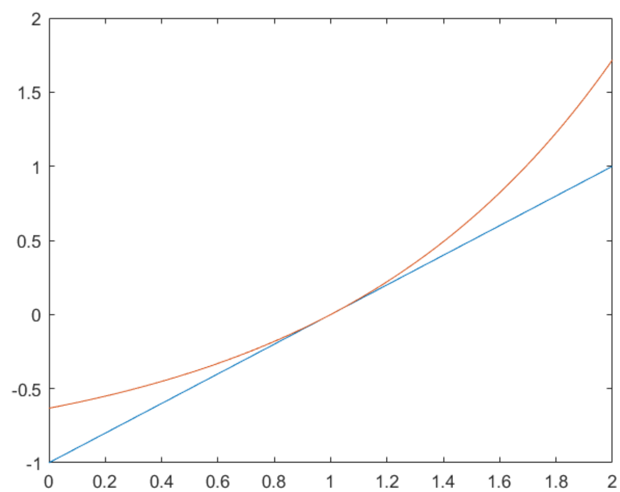


FIGURE 4. The graph of $f(x)$ (in red) and $g(x)$ (in blue)

(7) Use an appropriate theorem to complete the following statement.

If f is differentiable and $f(2) > f(6)$, then there is a number c, in the interval (___, ___) such that $f'(c)$?[</=/>] ___

What theorem guarantees this?

- The Mean Value Theorem
- The Increasing Function Theorem
- The Constant Function Theorem
- The Racetrack Principle

(Be sure that you can carefully apply this theorem to obtain the indicated result!)

Solution: Recall that

(a) The Increasing Function Theorem requires $f'(x) > 0$

(b) The Constant Function Theorem requires $f'(x) = 0$

(c) The Racetrack Principle requires two functions $f(x), g(x)$ and $f'(x) > g'(x)$

So the only theorem satisfying the condition of this problem is the Mean Value Theorem. According to the Mean Value Theorem, there exists $c \in (2,6)$, such that

$$f'(c) = \frac{f(6) - f(2)}{6 - 2} < 0$$

(8) Suppose $f(x)$ is continuous on $[3,8]$ and $-2 \leq f'(x) \leq 3$ for all x in $(3,8)$. Use the Mean Value Theorem to estimate $f(8)-f(3)$.

Answer: _____ $\leq f(8)-f(3) \leq$ _____

Solution: According to The Mean Value theorem. There exists $c \in (3,8)$ such that:

$$f'(c) = \frac{f(8)-f(3)}{8-3} = \frac{f(8)-f(3)}{5}$$

Since $-2 \leq f'(x) \leq 3$ for all x in $(3,8)$ it follows that

$$-2 \leq \frac{f(8)-f(3)}{5} \leq 3$$

$$-2 \cdot 5 \leq f(8)-f(3) \leq 3 \cdot 5$$

$$-10 \leq f(8)-f(3) \leq 15$$

(9) For the function $f(x) = x - \frac{6}{x}$, find all values of c in the interval [2,3] that satisfy the conclusion of the Mean-Value Theorem. If appropriate, leave your answer in radical form. Enter all fractions in lowest terms.

$c =$ ________

Solution:

If a function f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then by the Mean-Value Theorem there is at least one point c in (a,b) such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The function $f(x) = x - \frac{6}{x}$ satisfies the hypotheses of the Mean-Value Theorem on the interval [2,3], so there must exist a point c in (a,b) such that,

$$f'(c) = \frac{f(3) - f(2)}{3 - 2} = \frac{(3 - \frac{6}{3}) - (2 - \frac{6}{2})}{1} = 2.$$

$f'(x) = 1 + \frac{6}{x^2}$, so,

$$1 + \frac{6}{c^2} = 2 \quad \Rightarrow c = \sqrt{6}.$$

(10) Determine the intervals on which the given function is concave up or down and find the point of inflection. Let

$$f(x) = x\left(x - 6\sqrt{x}\right)$$

The x-coordinate of the point of inflection is ________

The **open** interval on the left of the inflection point is ________, and on this interval f is [?].

The **open** interval on the right is ________, and on this interval f is [?].

Solution: First, the domain of f is $[0, \infty)$

$$f(x) = x\left(x - 6\sqrt{x}\right) = x^2 - 6x^{3/2}$$

$$f'(x) = 2x - 9x^{1/2}$$

$$f''(x) = 2 - \frac{9}{2\sqrt{x}}$$

Let $f''(x) = 0$, we have $x = 5.0625$.

So the x-coordinate of the point of the inflection is $x = 5.0625$.

When $x < 5.0625$, $f''(x) < 0$, so f concave down on $(0, 5.0625)$.

When $x > 5.0625$, $f''(x) > 0$, so f concave up on $(5.0625, \infty)$.

 (11) Suppose that

$$f(x) = \frac{3e^x}{3e^x + 6}.$$

(A) Find all critical values of f. If there are no critical values, enter *None* . If there are more than one, enter them separated by commas.

Critical value(s) = ______

(B) Use **interval notation** to indicate where $f(x)$ is concave up.

Concave up: ____________

(C) Use **interval notation** to indicate where $f(x)$ is concave down.

Concave down: ____________

(D) Find all inflection points of f. If there are no inflection points, enter *None* . If there are more than one, enter them separated by commas.
Inflection point(s) at x = ______

Solution:

(a)

$$f'(x) = \frac{2e^x}{(e^x + 2)^2}$$

Obviously, for any $x \in \mathbb{R}$, $f'(x) > 0$, so there are no critical values.

(b)

$$f''(x) = \frac{2e^x(-e^x + 2)}{(e^x + 2)^3}$$

Let $f''(x) = 0$, we only need to solve the equation

$$-e^x + 2 = 0$$

then

$$e^x = 2$$

so $x = \ln(2)$
When $x < \ln(2)$, $f''(x) > 0$, so $f(x)$ concave up.
$x = \ln(2)$ is the inflection point.

(c) When $x > \ln(2)$, $f''(x) < 0$, so $f(x)$ concave down.

(12) Find the extreme values of the function f on the interval $[2,5]$. If an extreme value does not exist, enter **DNE** .

$$f(x) = x^8 + \frac{8}{x}$$

Absolute minimum value: ____
Absolute maximum value: ____
Solution: Set the derivative equal to zero to locate all critical numbers.

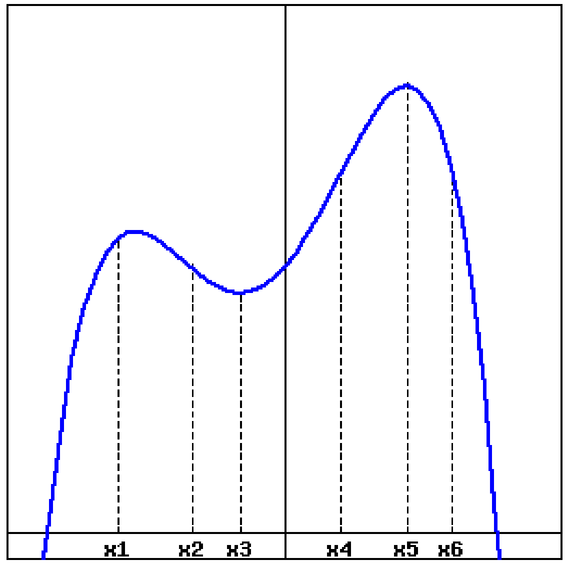
$$\begin{aligned} f'(x) = 8x^7 - \frac{8}{x^2} &= 0 \\ x^7 &= \frac{1}{x^2} \\ x^9 &= 1 \\ x &= 1 \end{aligned}$$

The only critical numbers is $x = 1$ Find the value of f at this critical number and the endpoints:

$$\begin{aligned} f(1) &= 9 \\ f(2) &= 260 \\ f(5) &= 390626.6 \end{aligned}$$

The absolute minimum value is 260, and the absolute maximum value is 390626.6.

[14] (13) The graph of f' (**not** f) is given below.



(Note that this is a graph of f', not a graph of f.)

Among the marked x-coordinates,

A. $f(x)$ is the greatest at $x =$ ____

B. $f(x)$ is the least at $x =$ ____

C. $f'(x)$ is the greatest at $x =$ ____

D. $f'(x)$ is the least at $x =$ ____

E. $f''(x)$ is the greatest at $x =$ ____

F. $f''(x)$ is the least at $x =$ ____

Solution:

A,B: Since f' is everywhere positive, f is everywhere increasing. Hence the greatest value of f is at x6 and the least value of f is at x1.

C,D: Directly from the graph, we see that f' is greatest at x5 and least at x3.

E,F: Since f'' gives the slope of the graph of f', f'' is greatest where f' is rising most rapidly, namely at x4, and f'' is least where f' is falling most rapidly, namely at x6.

(14) Please answer the following questions about the function

$$f(x) = \frac{6x^2}{x^2 - 4}.$$

Instructions:

- If you are asked for a function, enter a function.
- If you are asked to find x- or y-values, enter either a number or a list of numbers separated by commas. If there are no solutions, enter *None* .
- If you are asked to find an interval or union of intervals, use **interval notation**. Enter { } if an interval is empty.
- If you are asked to find a limit, enter either a number, I for ∞, -I for $-\infty$, or *DNE* if the limit does not exist.

(a) Calculate the first derivative of f. Find the critical numbers of f, where it is increasing and decreasing, and its local extrema.

$f'(x) =$ ____________________
Critical numbers $x =$ ____________________
Intervals where $f(x)$ is increasing ____________________ (Pay attention to endpoints. Separate multiple intervals with commas.)
Intervals where $f(x)$ is decreasing ____________________ (Pay attention to endpoints. Separate multiple intervals with commas.)
Local maxima $x =$ ____________________
Local minima $x =$ ____________________

(b) Find the following left- and right-hand limits at the vertical asymptote $x = -2$.

$$\lim_{x \to -2^-} \frac{6x^2}{x^2 - 4} = \boxed{?} \qquad \lim_{x \to -2^+} \frac{6x^2}{x^2 - 4} = \boxed{?}$$

Find the following left- and right-hand limits at the vertical asymptote $x = 2$.

$$\lim_{x \to 2^-} \frac{6x^2}{x^2 - 4} = \boxed{?} \qquad \lim_{x \to 2^+} \frac{6x^2}{x^2 - 4} = \boxed{?}$$

Find the following limits at infinity to determine any horizontal asymptotes.

$$\lim_{x \to -\infty} \frac{6x^2}{x^2 - 4} = \boxed{?} \qquad \lim_{x \to +\infty} \frac{6x^2}{x^2 - 4} = \boxed{?}$$

(c) Calculate the second derivative of f. Find where f is concave up, concave down, and has inflection points.

$f''(x) =$ ____________________
Open intervals where $f(x)$ is concave up ____________________
Open intervals where $f(x)$ is concave down ____________________
Inflection points $x =$ ____________________

(d) The function f is [?] because [?] for all x in the domain of f, and therefore its graph is symmetric about the [?]

(e) Answer the following questions about the function f and its graph.
The domain of f is the set (in **interval notation**) ____________________
The range of f is the set (in **interval notation**) ____________________
y-intercept ____________________
x-intercepts ____________________

(f) Sketch a graph of the function f without having a graphing calculator do it for you. Plot the y-intercept and the x-intercepts, if they are known. Draw dashed lines for horizontal and vertical asymptotes. Plot the points where f has local maxima, local minima, and inflection points. Use what you know from parts (a) - (c) to sketch the remaining parts of the graph of f. Use any symmetry from part (d) to your advantage. Sketching graphs is an important skill that takes practice, and you may be asked to do it on quizzes or exams.

Solution:

(a) The domain of $f(x)$ is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

$$f'(x) = \frac{-48x}{(x^2-4)^2}$$

So the critical number is $x = 0$. $f(x)$ is increasing on $(-\infty,-2),(-2,0)$, decreasing on $(0,2),(2,\infty)$. Local maximum $x = 0$, local minimum none.

(b)

$$\lim_{x\to-2^-}\frac{6x^2}{x^2-4} = \infty$$

$$\lim_{x\to-2^+}\frac{6x^2}{x^2-4} = -\infty$$

$$\lim_{x\to2^-}\frac{6x^2}{x^2-4} = -\infty$$

$$\lim_{x\to2^+}\frac{6x^2}{x^2-4} = \infty$$

$$\lim_{x\to-\infty}\frac{6x^2}{x^2-4} = 6$$

$$\lim_{x\to\infty} \frac{6x^2}{x^2-4} = 6$$

(c)

$$f''(x) = \frac{48(3x^2+4)}{(x^2-4)^3}$$

so $f''(x) > 0$ when $x < -2$ or $x > 2$, $f''(x) < 0$ when $-2 < x < 2$. Since $f(x)$ is undefined on $x = 2$ and $x = -2$, there is no inflection value. f concave up on $(-\infty,-2),(2,\infty)$, f concave down on $(-2,2)$.

(d) Since $f(x) = f(-x)$ for all x in the domain of f, f is even. It is symmetric about $x = 0$.

(e) The domain is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$. The range of f is $(-\infty,0]\cup(6,\infty)$,since $x^2-4 \in [-4,0)\cup(0,\infty)$. The x-intercept and y-intercept are both 0.

(f) The graph of f:

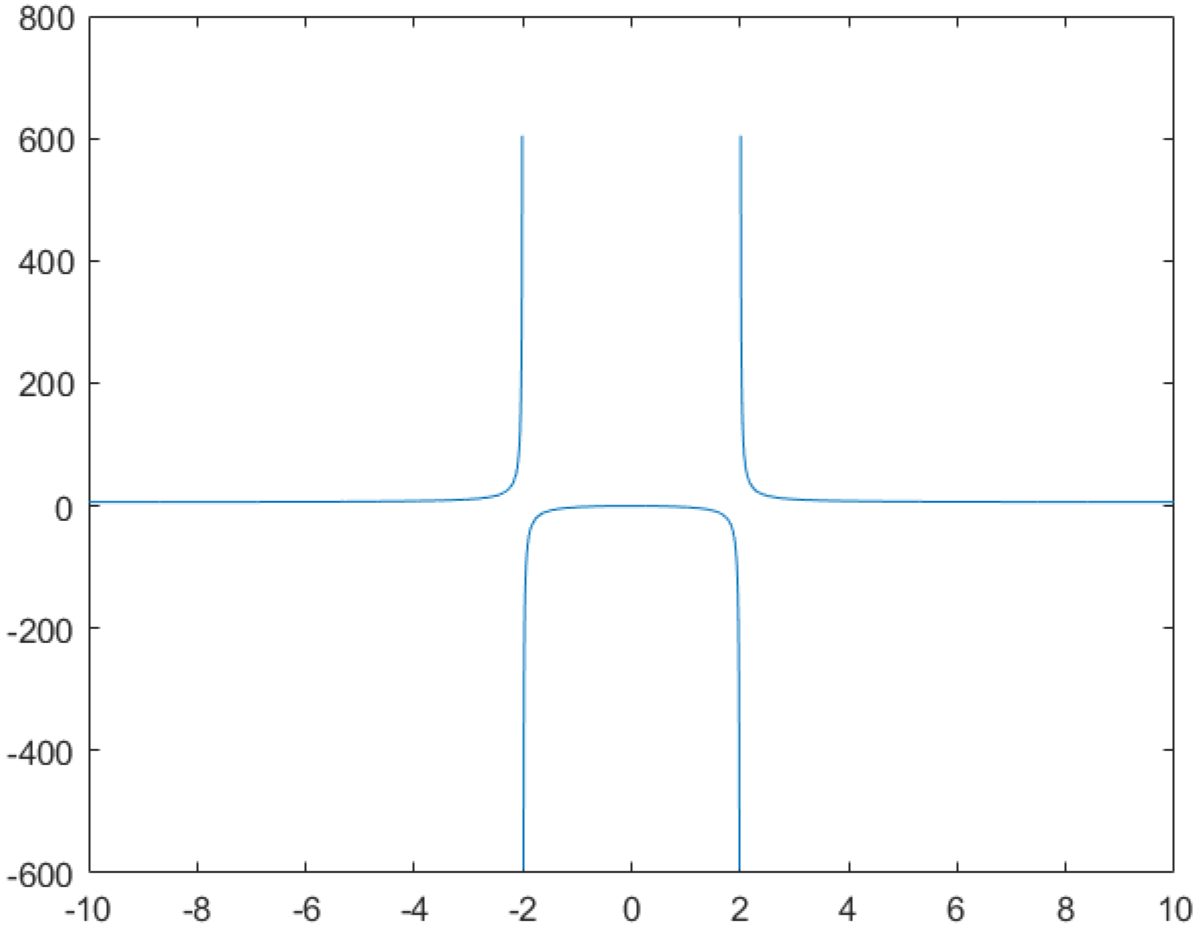


FIGURE 5. $f(x) = \frac{6x^2}{x^2-4}$

 (15) Suppose that

$$f(x) = (x^2 + 11)(1 - x^2).$$

(A) Find all critical points of f. If there are no critical points, enter None. If there are more than one, enter them separated by commas.
Critical point(s) = ______

(B) Find all intervals (separated by commas if more than one) where $f(x)$ is increasing. Pay attention to endpoints!

Note: When using interval notation in WeBWorK, you use **'inf'** for ∞, **'-inf'** for $-\infty$, and **'U'** for the union symbol. If there are no values that satisfy the required condition, then enter "" without the quotation marks.

Increasing:

____

(C) Find all intervals (separated by commas if more than one) where $f(x)$ is decreasing. Pay attention to endpoints!
Decreasing:

____

(D) Find the x-coordinates of all local maxima of f. If there are no local maxima, enter None. If there are more than one, enter them separated by commas.

Local maxima at x = _____

(E) Find the x-coordinates of all local minima of f. If there are no local minima, enter None. If there are more than one, enter them separated by commas.

Local minima at x = _____

(F) Find all open intervals where $f(x)$ is concave down.
Concave down:

____

Solution:

(a)
$$f'(x) = -4x(x^2 + 5)$$
Suppose $f'(x) = 0$, the only critical point is $x = 0$.

(b) When $x < 0$, $f'(x) > 0$. When $x > 0$, $f'(x) < 0$ so f is increasing on $(-\infty, 0]$.

(c) f is decreasing on $[0, \infty)$.

(d) Since f is increasing on $(-\infty, 0]$, decreasing on $[0, \infty)$, so $x = 0$ is the only maximum.

(e) According to the previous analysis, there is no local minimum.

(f)
$$f''(x) = -12x^2 - 20$$
So $f''(x) < 0$ for all $x \in \mathbb{R}$. So f is concave down on $(-\infty, \infty)$.

(16) Let $f(x) = \frac{(x+4)^2}{(x-4)^2}$.

Answer the following questions (for multiple answers enter each separated by commas e.g (a) 0,2 or (c) (-2,3),(0,-4) if no value enter ”none”.

(a) Vertical Asymptotes $x =$ __________

(b) Horizontal Asymptotes $y =$ __________

(c) Points where the graph crosses a horizontal asymptote $(x, y) =$ __________

(d) Critical Points $(x, y) =$ __________

(e) Inflection Points $(x, y) =$ __________

Solution:

(a) $f(x) = \frac{(x+4)^2}{(x-4)^2}$ has zero denominator and hence a vertical asymptote when $x = 4$.

(b) $\lim_{x \to \pm\infty} \frac{(x+4)^2}{(x-4)^2} = 1$ so there is a horizontal asymptote at $y = 1$.

(c) $\frac{(x+4)^2}{(x-4)^2} = 1$ only when $x = 0$ so that $(0, 1)$ is the only point where the graph crosses the horizontal asymptote.

(d) $f'(x) = \frac{-16(x+4)}{(x-4)^3} = 0$ gives the only Critical Point $(-4, 0)$.

(e) $f''(x) = \frac{32(x+8)}{(x-4)^4} = 0$ when $x = -8$ so $\left(-8, \frac{1}{9}\right)$ is an inflection point.

 (17) **NOTE:** When using interval notation in WeBWorK, remember that:
You use 'INF' for ∞ and '-INF' for $-\infty$.
And use 'U' for the union symbol.
Enter **DNE** if an answer does not exist.

$$f(x) = \frac{x^4}{4} - 2x^3 - 4$$

a) Determine the intervals on which f is concave up and concave down.
f is concave up on: ______
f is concave down on: ______

b) Based on your answer to part (a), determine the inflection points of f. Each point should be entered as an **ordered pair** (that is, in the form (x,y)).
______ (Separate multiple answers by commas.)

c) Find the critical numbers of f and use the Second Derivative Test, when possible, to determine the relative extrema. List only the x-coordinates.
Relative maxima at: ______ (Separate multiple answers by commas.)
Relative minima at: ______ (Separate multiple answers by commas.)

d) Find the x-value(s) where $f'(x)$ has a relative maximum or minimum.
f' has relative maxima at: ______ (Separate multiple answers by commas.)
f' has relative minima at: ______ (Separate multiple answers by commas.)

Solution:
(a)$f'(x) = x^3 - 6x^2$ and $f''(x) = 3x^2 - 12x$. Set $f''(x) = 0$ and solve.

$$3x^2 - 12x = 0 \Rightarrow 3x(x-4) = 0.$$

Thus, concavity could change at $x = 0$ and $x = 4$. By testing sample points in each interval, we find $f'' > 0$ (f is concave up) on $(-\infty, 0) \cup (4, \infty)$ and $f'' < 0$ (f is concave down) on $(0, 4)$.

(b)Therefore, there are inflection points at $x = 0, 4$. Plug these x values into the original function to find the corresponding y-values:

$$f(0) = -4, \quad f(4) = -68.$$

The inflection points occur at $(0, -4)$ and $(4, -68)$.

(c)Find critical numbers by setting $f'(x) = 0$

$$\begin{aligned} x^3 - 6x^2 &= 0 \\ x^2(x-6) &= 0 \\ \\ x &= 0, 6. \end{aligned}$$

Since $f''(0) = 0$, the Second Derivative Test is inconclusive. However, the First Derivative Test shows that there is neither a minimum nor maximum at $x = 0$, since $f'(x)$ does not change sign at $x = 0$.
Since $f''(6) > 0$, there is a local minimum at $x = 6$.

(d)To find where $f'(x)$ will have a relative extreme point, first set $f''(x) = 0$. As above, $x = 0, 4$.
Find $f'''(x) = 6x - 12$. Since $f'''(0) = -12 < 0$, there is a local maximum (for f') at $x = 0$. Since $f'''(4) = 12 > 0$, there is a local minimum (for f') at $x = 4$.

(18) A function $f(x)$ and interval $[a,b]$ are given. Check if the Mean Value Theorem can be applied to f on $[a,b]$. If so, find all values c in $[a,b]$ guaranteed by the Mean Value Theorem

Note, if the Mean Value Theorem does not apply, enter **DNE** for the c value.

$$f(x) = 3\sin^{-1} x \qquad \text{on } [-1,1]$$

$c =$ ________ (Separate multiple answers by commas.)

Solution:

$f(x) = 3\sin^{-1} x$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$, so the Mean Value Theorem applies.

$$\begin{aligned} f'(c) &= \frac{f(b)-f(a)}{b-a} \\ \frac{3}{\sqrt{1-c^2}} &= \frac{3\sin^{-1}(1) - 3\sin^{-1}(-1)}{1-(-1)} \\ \frac{3}{\sqrt{1-c^2}} &= \frac{3\pi}{2} \\ \sqrt{1-c^2} &= \frac{2}{\pi} \\ c &= \pm\sqrt{1-\frac{4}{\pi^2}} \end{aligned}$$

Both values of c are in the interval $(-1,1)$.

 (19) Suppose that

$$f(x) = x - 5x^{1/5}$$

(A) Find all critical values of f. If there are no critical values, enter -1000. If there are more than one, enter them separated by commas.
Critical value(s) = ______

(B) Use interval notation to indicate where $f(x)$ is increasing.

Note: When using interval notation in WeBWorK, you use **I** for ∞, **-I** for $-\infty$, and **U** for the union symbol. If there are no values that satisfy the required condition, then enter"" without the quotation marks.

Increasing:

______

(C) Use interval notation to indicate where $f(x)$ is decreasing.
Decreasing:

______

(D) Find the x-coordinates of all local maxima of f. If there are no local maxima, enter -1000. If there are more than one, enter them separated by commas.

Local maxima at x = ______

(E) Find the x-coordinates of all local minima of f. If there are no local minima, enter -1000. If there are more than one, enter them separated by commas.

Local minima at x = ______

(F) Use interval notation to indicate where $f(x)$ is concave up.
Concave up:

______

(G) Use interval notation to indicate where $f(x)$ is concave down.
Concave down:

______

(H) Find all inflection points of f. If there are no inflection points, enter -1000. If there are more than one, enter them separated by commas.
Inflection point(s) at x = ______

(I) Use all of the preceding information to sketch a graph of f. When you're finished, enter a "1" in the box below.

Graph Complete: ______

Solution:

(a)$f'(x) = 1 - x^{-\frac{4}{5}}$ and $f''(x) = \frac{4}{5}x^{-\frac{9}{5}}$. Set $f'(x) = 0$ and solve. We have

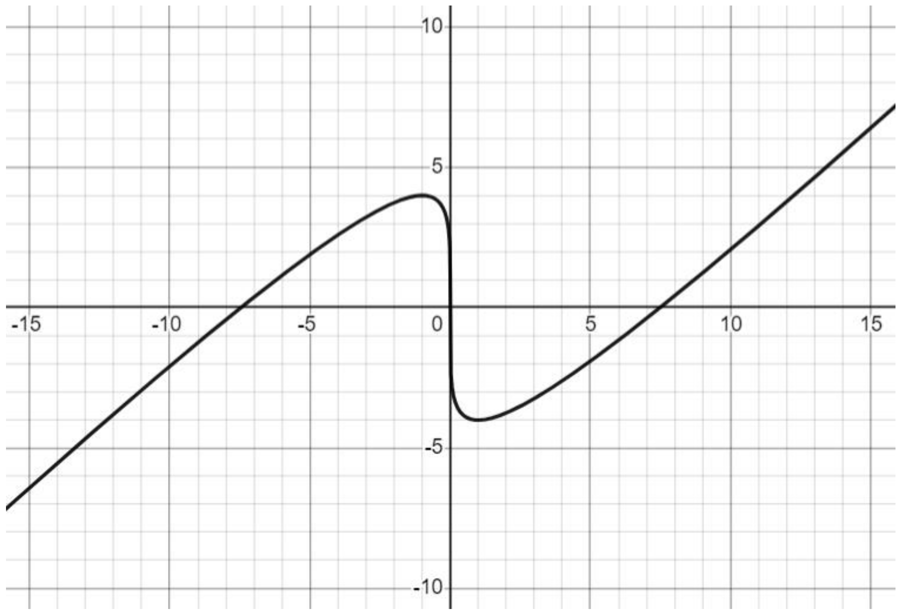
$$f'(x) = 1 - x^{-\frac{4}{5}} = 0 \Rightarrow x = \pm 1.$$

Then the critical values are $f(1) = -4$ and $f(-1) = 4$.

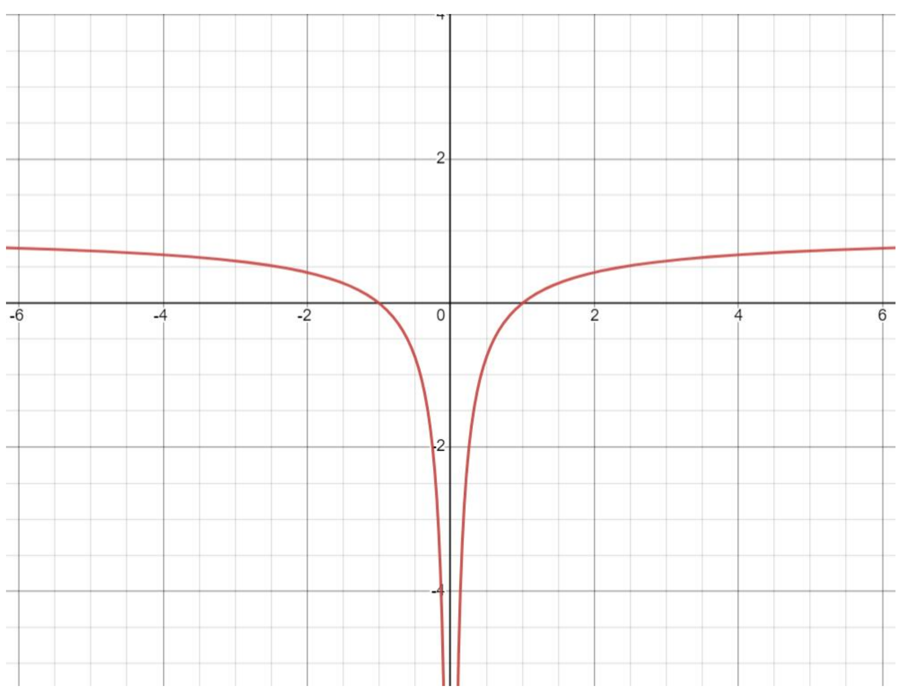
(b) We find $f'(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$. So f is increasing on $(-\infty, -1) \cup (1, \infty)$.

(c) $f'(x) < 0$ on$(-1, 1)$. So f is decreasing on $(-1, 1)$.

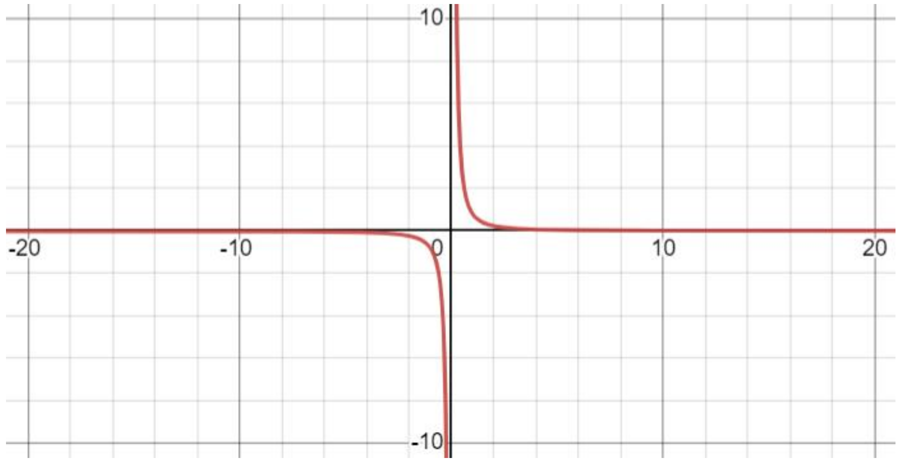
(d) By (b)(c), local maxima at $x = -1$.
(e) By (b)(c), local minima at $x = 1$.
(f) We find $f''(x) > 0$ on $x > 0$.
(g) $f''(x) < 0$ on $x < 0$.
(h) By (f)(g), the inflection point at $x = 0$.
(i)



$y = f(x)$



$y = f'(x)$



$y = f''(x)$

(20) Plot a graph of the rational function $f(x) = \frac{x^2}{x^2-64}$ and label the coordinates of the stationary points and inflection points. Show the horizontal and vertical asymptotes and label them with their equations. Label point(s), if any, where the graph crosses a horizontal asymptote. Check your work with a graphing utility.

Enter the following information from your graph (for multiple answers enter each separated by commas e.g (a) 0,2 or (c) (-2,3),(0,-4) if no value enter "none".

(a) Vertical Asymptotes $x =$ __________

(b) Horizontal Asymptotes $y =$ __________

(c) Points where the graph crosses a horizontal asymptote $(x, y) =$ __________

(d) Stationary Points $(x, y) =$ __________

(e) Inflection Points $(x, y) =$ __________

Solution:

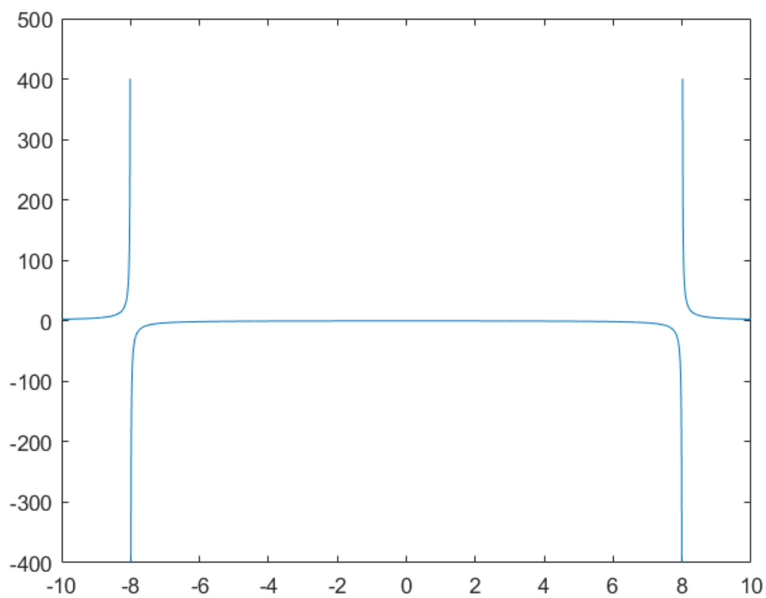
(a) $x = 0$

(b) $y = \pm 8$

(c) none

(d) $(0, 0)$

(e) none



$y = f(x)$