THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2022-2023 Term 1
Suggested Solutions of WeBWork Coursework 4

(1) Let

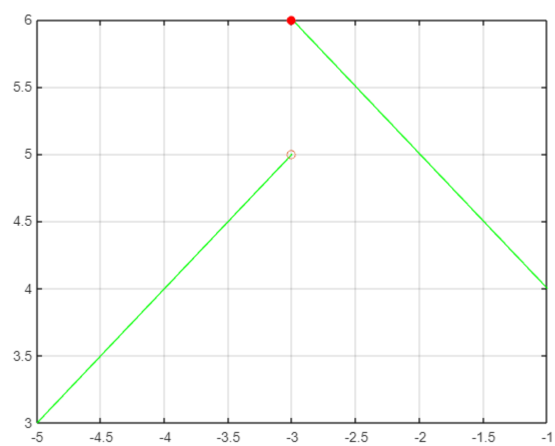
$$f(x) = \begin{cases} 8 + x, x < -3, \\ 3 - x, x \geq -3. \end{cases}$$

Find the indicated one-sided limits of f, and determine the continuity of f at the indicated point. You should also sketch a graph of $y = f(x)$, including hollow and solid circles in the appropriate places.
NOTE: Type DNE if a limit does not exist.

$\lim_{x \to -3^-} f(x) =$ ______________
$\lim_{x \to -3^+} f(x) =$ ______________
$\lim_{x \to -3} f(x) =$ ______________
$f(-3) =$ ______________

Is f continuous at -3?
(YES/NO) ____________

Solution:



$$\lim_{x \to -3^-} f(x) = 5$$

$$\lim_{x \to -3^+} f(x) = 6$$

$$\lim_{x \to -3} f(x) = DNE$$

$$f(-3) = 6$$

Is f continuous at -3?
No, since the limit of f at -3 does not exist.

(2) Let

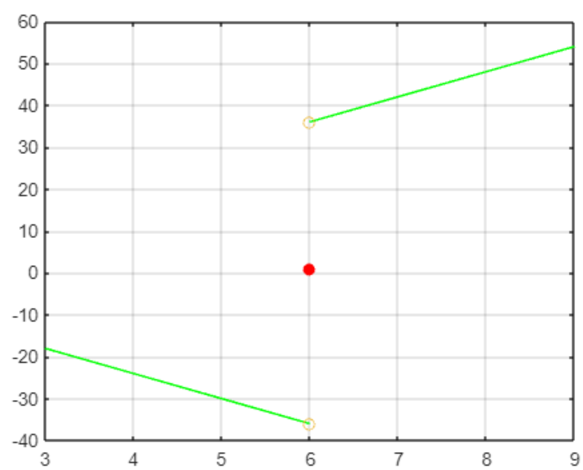
$$f(x) = \begin{cases} -6x, & x < 6, \\ 1, & x = 6, \\ 6x, & x > 6. \end{cases}$$

Find the indicated one-sided limits of f, and determine the continuity of f at the indicated point. You should also sketch a graph of $y = f(x)$, including hollow and solid circles in the appropriate places.
NOTE: Type DNE if a limit does not exist.

$\lim_{x\to 6^-} f(x) =$ ____________
$\lim_{x\to 6^+} f(x) =$ ____________
$\lim_{x\to 6} f(x) =$ ____________
$f(6) =$ ____________

Is f continuous at $x = 6$?
(YES/NO) __________

Solution:



$$\lim_{x\to 6^-} f(x) = -36$$

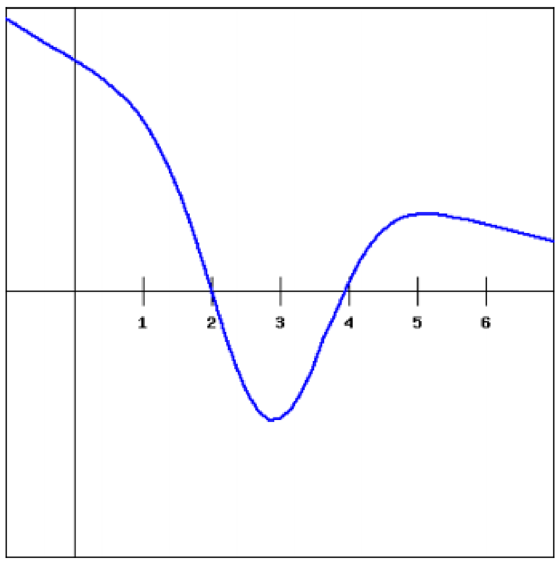
$$\lim_{x\to 6^+} f(x) = 36$$

$$\lim_{x\to 6} f(x) = DNE$$

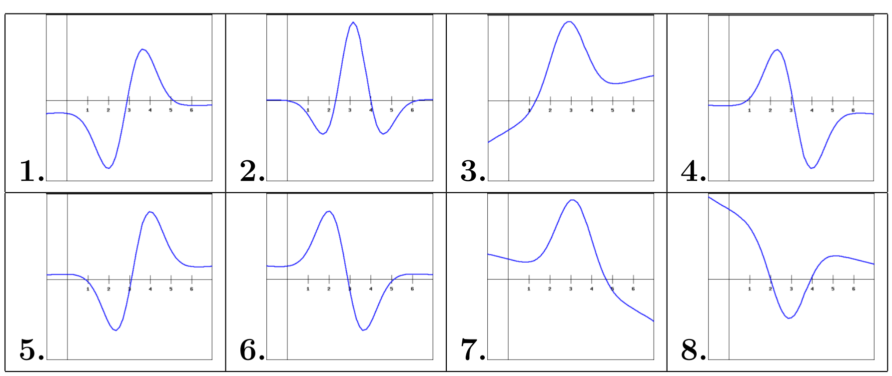
$$f(6) = 1$$

Is f continuous at $x = 6$?
No, since the limit of f at 6 does not exist.

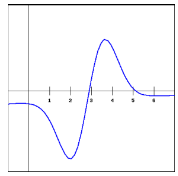
(3) For the function $f(x)$ shown in the graph below, sketch a graph of the derivative. You will then be picking which of the following is the correct derivative graph, but should be sure to first sketch the derivative yourself.



Which of the following graphs is the derivative of $f(x)$? [?/1/2/3/4/5/6/7/8]
(Click on a graph to enlarge it.)



Solution:
Because the derivative gives the slope of the original function at each point x, we know that the derivative is negative where $f(x)$ is decreasing and positive where it is increasing. Applying this to $f(x)$, we see that the derivative must be



which is answer 1.

(4) Let $f(x) = |x - 2|$. Evaluate the following limits.

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \_\_\_\_\_$$

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \_\_\_\_\_$$

Thus the function $f(x)$ is not differentiable at 2.

Solution:
Recall that

$$|x - 2| = \begin{cases} -(x-2), x < 2 \\ x - 2, x \geq 2 \end{cases}.$$

Hence

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{-(x-2) - 0}{x - 2} = \lim_{x\to 2^-} (-1) = -1$$

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{(x-2) - 0}{x - 2} = \lim_{x\to 2^+} (1) = 1$$

(5) Let

$$f(x) = \begin{cases} -(2x+1) & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

Find a formula for $f'(x)$.

$$f'(x) = \begin{matrix} \_\_\_\_\_\_\_\_\_ & \text{if} & x < -1 \\ \_\_\_\_\_\_\_\_\_ & \text{if} & -1 \leq x < 1 \\ \_\_\_\_\_\_\_\_\_ & \text{if} & x > 1 \end{matrix}$$

Solution:

$$\lim_{x\to(-1)^-}\frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to(-1)^-}\frac{-(2x+1)-1}{x+1} = -2,$$
$$\lim_{x\to(-1)^+}\frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to(-1)^+}\frac{x^2-1}{x+1} = \lim_{x\to(-1)^+}(x-1) = -2.$$

Hence f is differentiable at $x=-1$ with $f'(-1)=-2$. However, f is not differentiable at $x=1$ as

$$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+}\frac{x-1}{x-1} = 1, \text{ while}$$
$$\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-}\frac{x^2-1}{x-1} = \lim_{x\to1^-} x+1 = 2.$$

Therefore, we have

$$f'(x) = \begin{cases} -2 & \text{if } x \le -1 \\ 2x & \text{if } -1 < x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

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(6) **Part 1: The derivative at a specific point**
Use the definition of the derivative to compute the derivative of $f(x)=\sqrt{x+7}$ at the specific point $x=2$. Evaluate the limit by using algebra to simplify the difference quotient (in first answer box) and then evaluating the limit (in the second answer box).

$$f'(2) = \lim_{h\to0}\left(\frac{f(2+h)-f(2)}{h}\right) = \lim_{h\to0}\left(\ \_\_\_\_\_\_\_\_\ \right) = \_\_\_.$$

Part 2: The derivative function
Part 3: The tangent line

Solution:

$$f'(2) = \lim_{h\to0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to0}\frac{(\sqrt{2+h+7}-\sqrt{2+7})}{h}$$
$$= \lim_{h\to0}\frac{(\sqrt{2+h+7}-\sqrt{2+7})(\sqrt{2+h+7}+\sqrt{2+7})}{h(\sqrt{2+h+7}+\sqrt{2+7})}$$
$$= \lim_{h\to0}\frac{1}{\sqrt{2+h+7}+\sqrt{2+7}} = \frac{1}{2\sqrt{2+7}} = \frac{1}{6}.$$
$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to0}\frac{(\sqrt{x+h+7}-\sqrt{x+7})(\sqrt{x+h+7}+\sqrt{x+7})}{h(\sqrt{x+h+7}+\sqrt{x+7})}$$
$$= \lim_{h\to0}\frac{1}{\sqrt{x+h+7}+\sqrt{x+7}} = \frac{1}{2\sqrt{x+7}}.$$

The tangent line to f at $x = 2$ passes through the point $(2, f(2)) = (2, 3)$ on the graph of f and the equation for the tangent line to f at $x = 2$ is $y = 3 + \frac{1}{6}(x - 2)$.

(7) A function $f(x)$ is said to have a **jump** discontinuity at $x = a$ if :

1. $\lim_{x \to a^-} f(x)$ exists.
2. $\lim_{x \to a^+} f(x)$ exists.
3. The left and right limits are not equal.

Let $f(x) = \begin{cases} 3x - 8, & \text{if } x < 4 \\ \frac{5}{x+9}, & \text{if } x \geq 4 \end{cases}$

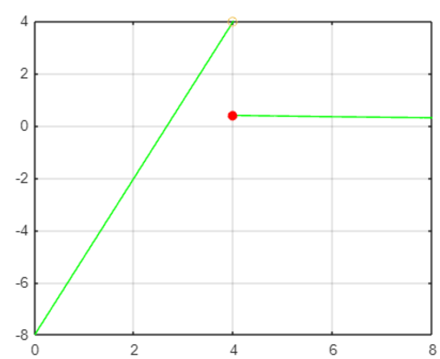
Show that $f(x)$ has a jump discontinuity at $x = 4$ by calculating the limits from the left and right at $x = 4$.

$\lim_{x \to 4^-} f(x) =$_______

$\lim_{x \to 4^+} f(x) =$_______

Now, for fun, try to graph $f(x)$.

Solution:



We have $\lim_{x \to 4^-} f(x) = 4$ while $\lim_{x \to 4^+} f(x) = \dfrac{5}{13}$.

(8) Find $f'(x)$ and $f'(0)$ where:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Find the derivative of $f(x)$ for x not equal 0.
$f'(x) =$ ___________
(b) If the derivative does not exist enter DNE.
$f'(0) =$ ___________

Solution:

Using the definition of the derivative we find that:
$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} h^2 \sin\left(\frac{1}{h}\right)\frac{1}{h} = \lim_{h\to 0} h\sin(\frac{1}{h}) = 0,$$
by the Squeeze Theorem.

For $x \neq 0$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 \sin\frac{1}{x+h} - x^2\sin\frac{1}{x}}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left(x^2(\sin\frac{1}{x+h} - \sin\frac{1}{x}) + 2hx\sin\frac{1}{x+h} + h^2\sin\frac{1}{x+h}\right).$$

The second term will tend to $2x\sin\frac{1}{x}$ and the third term will tend to 0 by the Squeeze Theorem. For the treatment of the first term, recall the facts that

$$\sin a - \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2} \text{ and } \lim_{x\to 0}\frac{\sin x}{x} = 1 .$$

By using these facts, we have, for $x \neq 0$,

$$\lim_{h\to 0}\frac{x^2}{h}(\sin\frac{1}{x+h} - \sin\frac{1}{x}) = \lim_{h\to 0}\frac{2x^2}{h}\cos\frac{2x+h}{2x(x+h)}\sin\frac{-h}{2x(x+h)}$$
$$= \cos\frac{1}{x}\lim_{h\to 0}\frac{2x^2}{h}\sin\frac{-h}{2x(x+h)}$$
$$= \cos\frac{1}{x}\lim_{h\to 0}\left(\frac{2x^2}{-2x(x+h)}\frac{\sin\frac{-h}{2x(x+h)}}{\frac{-h}{2x(x+h)}}\right)$$
$$= (\cos\frac{1}{x})(-1)(1) = -\cos\frac{1}{x}.$$

Therefore we have

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \text{ for } x \neq 0.$$

(9) Let

$$f(x) = \begin{cases} -6x^2+6x & \text{for } x<0, \\ 4x^2-3 & \text{for } x \geqslant 0. \end{cases}$$

According to the definition of the derivative, to compute $f'(0)$, we need to compute the left-hand limit

$\lim_{h\to 0^-}$ ____________, which is _____,

and the right-hand limit

$\lim_{h\to 0^+}$ ____________, which is _____.

We conclude that $f'(0)$ is _____.

Note: If a limit or derivative does not exist, and is not $\pm\infty$, enter 'DNE' as your answer. Enter 'inf' for ∞, '-inf' for $-\infty$.

Solution:

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-6h^2+6h+3}{h} = \lim_{h\to 0^-} (-6h+6+\frac{3}{h}) = -\infty.$$

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{4h^2-3+3}{h} = \lim_{h\to 0^+} (4h) = 0.$$

Hence f is not differentiable at $x=0$.

(10) Evaluate the limit.

$$\lim_{x\to -\infty} \frac{x^3+5x^2-x+6}{3-2x^2}$$

Enter **INF** for ∞, **-INF** for $-\infty$, or **DNE** if the limit does not exist, but is neither ∞ nor $-\infty$.
Limit: ____

Solution:
Note that the numerator has a higher degree than the denominator. For very large **negative** values of x,

$$\frac{x^3+5x^2-x+6}{3-2x^2} \approx \frac{x^3}{-2x^2} = \frac{x}{-2} \to \infty.$$

(11) Evaluate the following limits. If needed, enter 'INF' for ∞ and '-INF for $-\infty$.

(a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{9+10x^2}}{2+9x} =$ ______

(b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{9+10x^2}}{2+9x} =$ ______

Solution:

$$\lim_{x\to\infty} \frac{\sqrt{9+10x^2}}{2+9x} = \lim_{x\to\infty} \frac{\sqrt{\frac{9}{x^2}+10}}{\frac{2}{x}+9} = \lim_{t\to 0^+} \frac{\sqrt{9t^2+10}}{2t+9} = \frac{\sqrt{10}}{9}$$

$$\lim_{x\to-\infty} \frac{\sqrt{9+10x^2}}{2+9x} = \lim_{x\to-\infty} \frac{\sqrt{\frac{9}{x^2}+10}}{-\frac{2}{x}-9} = \lim_{t\to 0^-} \frac{\sqrt{9t^2+10}}{-2t-9} = -\frac{\sqrt{10}}{9}$$

(12) Carry out three steps of the Bisection Method for $f(x)=2^x-x^4$ as follows:
(a) Show that $f(x)$ has a zero in $[1,2]$.
(b) Determine which subinterval, $[1,1.5]$ or $[1.5,2]$, contains a zero.

(c) Determine which interval, $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, or $[1.75, 2]$, contains a zero.

In part (b), the interval with a zero is _________.

In part (c), the interval with a zero is _________.

Solution:

Note that $f(x)$ is continuous for all x.

(a) $f(1) = 1 > 0$, $f(2) = -12 < 0$. Hence $f(x) = 0$ for some x between 1 and 2.

(b) $f(1) > 0$ and $f(1.5) \approx -2.234 < 0$. Hence $f(x) = 0$ for some x between 1 and 1.5.

(c) $f(1.25) \approx -0.062 < 0$. Hence $f(x) = 0$ for some x between 1 and 1.25.

(13) Find a and b so that the function

$$f(x) = \begin{cases} 5x^3 - 3x^2 + 4, & x < -2, \\ ax + b, & x \geq -2 \end{cases}$$

is both continuous and differentiable.

$a =$ _________

$b =$ _________

Solution:

Since $f(x)$ is both continuous and differentiable, we have

$$\begin{cases} \lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^+} f(x) = f(-2) \\ f'_-(-2) = f'_+(-2) \end{cases}.$$

By solving the equation,

$$\begin{cases} \lim\limits_{x \to -2^-} f(x) = f(-2) \\ \lim\limits_{t \to -0^-} \frac{f(-2+t) - f(-2)}{t} = \lim\limits_{t \to 0^+} \frac{f(-2+t) - f(-2)}{t} \end{cases},$$

which is

$$\begin{cases} 5 \cdot (-2)^3 - 3 \cdot (-2)^2 + 4 = -2a + b \\ \lim\limits_{t \to -0^-} \frac{\left[5 \cdot (-2+t)^3 - 3 \cdot (-2+t)^2 + 4\right] - \left[5 \cdot (-2)^3 - 3 \cdot (-2)^2 + 4\right]}{t} = \lim\limits_{t \to 0^+} \frac{[(-2+t)a + b] - (-2a + b)}{t} \end{cases},$$

i.e.

$$\begin{cases} -48 = -2a + b \\ 3 \cdot 5 \cdot (-2)^2 - 2 \cdot 3 \cdot (-2)^1 = a \end{cases},$$

we obtain,

$$\begin{cases} a = 72 \\ b = 96 \end{cases}.$$

(14) Let $f(x) = \dfrac{3}{\sqrt[3]{x}}$. Evaluate each of the following:

1. $f'(2) =$ ____________
2. $f'(7) =$ ____________

Solution:

$$f'(x) = -\frac{3}{3}x^{-\frac{4}{3}} = -x^{\frac{4}{3}}$$

Then we have

$$f'(2) = -2^{-\frac{4}{3}} \approx -0.39685$$
$$f'(7) = -7^{-\frac{4}{3}} \approx -0.0746797$$

(15) Differentiate the following function:

$$u = \sqrt[3]{t^2} - 4\sqrt{t^3}$$

$u' =$ ____________

Solution:

$$u'(t) = \frac{2}{3}t^{-\frac{1}{3}} - 4 \cdot \frac{3}{2}t^{\frac{1}{2}} = \frac{2}{3}t^{-\frac{1}{3}} - 6t^{\frac{1}{2}}$$

(16) Differentiate the following function:

$$f(t) = \sqrt[4]{t} - \frac{1}{\sqrt[4]{t}}$$

$f'(t) =$ ________

Solution:

$$f'(t) = \frac{1}{4}t^{-\frac{3}{4}} + \frac{1}{4}t^{-\frac{5}{4}} = \frac{1}{4}\left(t^{-\frac{3}{4}} + t^{-\frac{5}{4}}\right)$$

(17) Suppose $f'(x)$ exists for all x in (a, b).

Mark all true items with a check. There may be more than one correct answer.

- A. $f(x)$ is continuous on (a, b).

- B. $f(x)$ is continuous at $x = a$.

- C. $f(x)$ is defined for all x in (a, b).

- D. $f'(x)$ is differentiable on (a, b).

Solution:

- A. Correct! Any function which is differentiable is continuous.
- B. Incorrect! Let $f(x) = \frac{1}{x-a}$.
- C. Correct! According to the definition of differentiable, it is obvious.
- D. Incorrect! Let $(a, b) = (-2, 1)$. Define

$$f(x) = \begin{cases} -(2x+1) & \text{when } x < -1 \\ x^2 & \text{when } x \geq -1 \end{cases}$$

Then

$$f'(x) = \begin{cases} -2 & \text{when } x < -1 \\ 2x & \text{when } x \geq -1 \end{cases}$$

which is not differentiable on $x = -1$.

(18) If $f'(a)$ exists, then $\lim_{x \to a} f(x)$

- A. must exist, but there is not enough information to determine its value.

- B. is equal to $f(a)$.

- C. is equal to $f'(a)$.

- D. might not exist.

- E. does not exist.

Solution:
Since $f'(a)$ exists, f is continuous on $x = a$. So $\lim_{x \to a} f(x) = f(a)$. So the answer is B.