THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2022-2023 Term 1
Suggested Solutions of WeBWork Coursework 1

(1) In each part, find a formula for the general term of the sequence, starting with $n = 1$.

(a) $1, 1, 1, 1, \ldots$

(b) $1, -1, 1, -1, \ldots$

(c) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16} \ldots$

(d) $0, \frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \ldots$

Solution:

(a) The general term of the sequence is $a_n = 1$.

(b) The general term of the sequence is $a_n = (-1)^{n+1}$.

(c) The general term of the sequence is $a_n = 1 - \frac{1}{2^n}$.

(d) The general term of the sequence is $a_n = \frac{(n-1)^2}{\sqrt[n]{\pi}}$.

(2) Determine whether the following sequences converge or diverge.

(a) $\{0, 5, 0, 0, 5, 0, 0, 0, 5, \ldots\}$

(b) $a_n = \frac{\sin 5n}{1 + \sqrt{n}}$

(c) $a_n = \frac{n^n}{n!}$

Solution:

(a) The sequence diverges because both 0 and 5 appear indefinitely in the tail of the sequence.

(b) Note that, for all $n \geq 1$, $-1 \leq \sin 5n \leq 1$, and hence

$$-\frac{1}{1+\sqrt{n}} \leq \frac{\sin 5n}{1+\sqrt{n}} \leq \frac{1}{1+\sqrt{n}}.$$

Also, we have $\lim_{n\to\infty} -\frac{1}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$.

By squeeze theorem, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin 5n}{1+\sqrt{n}} = 0$. The sequence converges.

(c) Note that, for all $n \geq 1$,

$$a_n = \frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{2} \cdots \frac{n}{n} \geq n \cdot 1 \cdots 1 = n.$$

The sequence is unbounded and hence diverges.

(3) Determine whether the sequence $a_n = \dfrac{n^2 + \sin(4n+14)}{n^4+14}$ converges or diverges. If it converges, find the limit.

Solution: Note that, for all $n \geq 1$, $-1 \leq \sin(4n+14) \leq 1$, and hence

$$-\frac{1}{n^4+14} \leq \frac{\sin(4n+14)}{n^4+14} \leq \frac{1}{n^4+14}.$$

Also, we have $\lim_{n\to\infty} -\dfrac{1}{n^4+14} = \lim_{n\to\infty} \dfrac{1}{n^4+14} = 0$.

By squeeze theorem, $\lim_{n\to\infty} \dfrac{\sin(4n+14)}{n^4+14} = 0$. Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^4+14} + \lim_{n\to\infty} \frac{\sin(4n+14)}{n^4+14} = \lim_{n\to\infty} \frac{1/n^2}{1+14/n^4} + 0 = 0.$$

(4) Use algebra to simplify the expression before evaluating the limit. In particular, factor the highest power of n from the numerator and denominator, then cancel as many factors of n as possible.

$$\lim_{n\to\infty} \frac{9n}{(8n^7+5)^{1/7}}$$

Solution:

$$\lim_{n\to\infty} \frac{9n}{(8n^7+5)^{1/7}} = \lim_{n\to\infty} \frac{9n}{n(8+5/n^7)^{1/7}} = \lim_{n\to\infty} \frac{9}{(8+5/n^7)^{1/7}} = \frac{9}{8^{1/7}}.$$

(5) Part 1: Evaluating a series

Consider the sequence $\{a_n\} = \left\{\dfrac{2}{n^2+2n}\right\}$.

(a) Find $\lim_{n\to\infty} a_n$ if it exists.

(b) Find $\sum_{n=1}^{\infty} a_n$ the sum of all terms in this sequence, which is defined as the limit of the partial sums, if it exists.

Solution:

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2}{n^2+2n} = \lim_{n\to\infty} \dfrac{2/n^2}{1+2/n} = \dfrac{0}{1+0} = 0.$

(b) Note that, for $n \geq 1$,

$$a_n = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

Hence, for $N \geq 2$,

$$\begin{aligned}
\sum_{n=1}^{N} a_n &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right) \\
&= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+2}\right) \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N}\right) - \frac{1}{N+1} - \frac{1}{N+2} \\
&= \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}.
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} \left(\frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right) = \frac{3}{2}.$$

Part 2: Evaluating another series

Consider the sequence $\{b_n\} = \left\{ \ln\left(\frac{n+1}{n}\right) \right\}$.

(a) Find $\lim\limits_{n\to\infty} b_n$ if it exists.

(b) Find $\sum\limits_{n=1}^{\infty} b_n$ if it exists.

Solution:

(a) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \lim\limits_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = \ln(1+0) = 0.$

(b) Note that, for $n \geq 1$,

$$b_n = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n.$$

Hence, for $N \geq 1$,

$$\begin{aligned}
\sum_{n=1}^{N} b_n &= \sum_{n=1}^{N} (\ln(n+1) - \ln n) \\
&= \sum_{n=1}^{N} \ln(n+1) - \sum_{n=1}^{N} \ln n \\
&= \sum_{n=2}^{N+1} \ln n - \sum_{n=1}^{N} \ln n \\
&= \ln(N+1) - \ln 1 \\
&= \ln(N+1)
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} b_n = \lim_{N\to\infty} \sum_{n=1}^{N} b_n = \lim_{N\to\infty} \ln(N+1) = +\infty.$$

Part 3: Developing conceptual understanding

Suppose $\{c_n\}$ is a sequence.

(a) If $\lim\limits_{n\to\infty} c_n = 0$, then the series $\sum\limits_{n=1}^{\infty} c_n$

- must
- may or may not
- cannot

converge.

(b) If $\lim\limits_{n\to\infty} c_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} c_n$

- must
- may or may not
- cannot

converge.

(c) If the series $\sum_{n=1}^{\infty} c_n$ converges, then $\lim_{n\to\infty} c_n$
- must
- may or may not
- cannot

be equal to 0.

Solution:

(a) If $\lim_{n\to\infty} c_n = 0$, then the series $\sum_{n=1}^{\infty} c_n$ **may or may not** converge. Just look back at parts 1 and 2.

(b) If $\lim_{n\to\infty} c_n \neq 0$, then the series $\sum_{n=1}^{\infty} c_n$ **cannot** converge.

(c) If the series $\sum_{n=1}^{\infty} c_n$ converges, then $\lim_{n\to\infty} c_n$ **must** be equal to 0.

Explanation. (a) Just see Part 1 $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n$ converges while for Part 2, $\lim_{n\to\infty} b_n = 0$ but $\sum_{n=1}^{\infty} b_n$ diverges.

(b) This is a result of (c) by the argument of contradiction: first claim that statement (c) holds, now suppose (b) is false, i.e. the series $\sum_{n=1}^{\infty} c_n$ converges, then according to (c) c_n must tend to zero, which contradicts the initial setting $\lim_{n\to\infty} c_n \neq 0$.

(c) Finally, we prove statement (c): Just set the partial sum

$$S_n = \sum_{k=1}^{n} c_k.$$

Then the series $\sum_{n=1}^{\infty} c_n$ converges is equivalent to $\lim_{n\to\infty} S_n = C < \infty$. Note that $\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = C$ by setting $S_0 = 0$, therefore

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = C - C = 0.$$

□

(6) Consider the recursively defined sequence:

$$a_1 = 6$$

$$a_{n+1} = \frac{n+1}{n^2} a_n, \qquad \text{for } n \geq 1$$

(a) The sequence is

- Eventually monotone increasing
- Eventually monotone decreasing
- Neither

(b) The sequence is bounded below by

(c) The sequence is bounded above by

(d) The limit of the sequence is

Solution:

(a) The sequence is eventually monotone decreasing since $a_2 = 12 > 6 = a_1$ while
$$a_{n+1} = \frac{n+1}{n^2} a_n \leq a_n \quad \text{for } n \geq 2.$$
(b) Clearly $a_n \geq 0$ for all $n \geq 1$. So the sequence is bounded below by 0.
(c) From (a) we see that $a_n \leq a_2 = 12$ for $n \geq 1$. So the sequence is bounded above by 12.
(d) By Monotone Convergence Theorem, $\{a_n\}$ converges to some limit ℓ. Thus
$$\begin{aligned}\ell = \lim_{n\to\infty} a_{n+1} &= \left(\lim_{n\to\infty} \frac{n+1}{n^2}\right)\left(\lim_{n\to\infty} a_n\right) \\ &= \left(\lim_{n\to\infty} (\frac{1}{n} + \frac{1}{n^2})\right)\ell \\ &= 0 \cdot \ell = 0.\end{aligned}$$
Therefore the limit of the sequence $\{a_n\}$ is 0.

(7) Find the following limit.
$$\lim_{n\to\infty} [e^{-5n} \sin(5n)]$$
Solution: Note that, for all $n \geq 1$, $-1 \leq \sin(5n) \leq 1$, $e^{-5n} > 0$, and hence
$$-e^{-5n} \leq e^{-5n} \sin(5n) \leq e^{-5n}.$$
Also, we have $\lim_{n\to\infty} -e^{-5n} = \lim_{n\to\infty} e^{-5n} = 0$.
By squeeze theorem, $\lim_{n\to\infty} [e^{-5n} \sin(5n)] = 0$.

(8) Consider the recursively defined sequence:
$$a_1 = \sqrt{3}$$
$$a_{n+1} = \sqrt{3 + a_n}, \quad \text{for } n \geq 1$$
(a) The sequence is

- Monotone increasing
- Monotone decreasing
- Neither

(b) The sequence is bounded below by

(c) The sequence is bounded above by

(d) The limit of the sequence is

Solution:

(a) The sequence is monotone increasing.
To see this let $Q(n)$ be the statement "$a_{n+1} \geq a_n$".

- When $n = 1$, $a_2 = \sqrt{3+\sqrt{3}} \geq \sqrt{3} = a_1$. Therefore $Q(1)$ is true.
- Suppose $Q(n)$ is true for some natural number $n \geq 1$, i.e. $a_{n+1} \geq a_n$. Then,

$$a_{n+2} \geq \sqrt{3+a_{n+1}} \geq \sqrt{3+a_n} = a_{n+1}.$$

  Therefore, $Q(n+1)$ is true.

By mathematical induction, $a_{n+1} \geq a_n$ for all natural numbers n. Hence $\{a_n\}$ is monotone increasing.

(b) The sequence is bounded below by 0 and bounded above by 3.
To see this, let $P(n)$ be the statement "$0 \leq a_n \leq 3$".

- When $n = 1$, $0 \leq a_1 = \sqrt{3} \leq 3$. Therefore $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number $n \geq 1$, i.e. $0 \leq a_n \leq 3$. Then,

$$0 \leq a_{n+1} = \sqrt{3+a_n} \leq \sqrt{3+3} \leq 3.$$

  Therefore, $P(n+1)$ is true.

By mathematical induction, $0 \leq a_n \leq 3$ for all natural numbers n. Hence $\{a_n\}$ is bounded.

(c) By Monotone Convergence Theorem, $\{a_n\}$ is convergent. Let $\lim_{n\to\infty} a_n = A$. By (b), A should satisfy $0 \leq A \leq 3$. Since $a_{n+1}^2 = 3 + a_n$, we have

$$\begin{aligned} \lim_{n\to\infty} a_{n+1}^2 &= \lim_{n\to\infty} (3+a_n) \\ A^2 &= 3 + A \\ A^2 - A - 3 &= 0. \end{aligned}$$

So $A = \dfrac{1+\sqrt{13}}{2}$ or $A = \dfrac{1-\sqrt{13}}{2}$, where the latter is rejected since $a_n \geq 0$.
Therefore, $\lim_{n\to\infty} a_n = \dfrac{1+\sqrt{13}}{2}$.

(9) Consider the recursively defined sequence:

$$a_1 = 1, \quad a_2 = 1$$

$$a_{n+2} = \frac{a_{n+1}+a_n}{2}, \qquad \text{for } n \geq 1$$

Find the limit of the sequence if it exists.

Solution:

From the definition of the sequence,

$$a_3 = \frac{a_2+a_1}{2} = \frac{1+1}{2} = 1,$$

$$a_4 = \frac{a_3+a_2}{2} = \frac{1+1}{2} = 1,$$

and so on, we thus have

$$a_{n+2} = \frac{a_{n+1}+a_n}{2} = \frac{1+1}{2} = 1, \qquad \text{for } n \geq 1.$$

Hence the sequence is just a constant sequence of 1's, and clearly $\lim_{n\to\infty} a_n = 1$.

(10) Consider the sequence

$$a_n = \frac{n\cos(n\pi)}{2n-1}.$$

Write the first five terms of a_n, and find $\lim_{n\to\infty} a_n$.

Solution: The first five terms are

$$a_1 = -1,\ a_2 = \frac{2}{3},\ a_3 = -\frac{3}{5},\ a_4 = \frac{4}{7},\ a_5 = -\frac{5}{9}.$$

Note that

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{2n\cos(2n\pi)}{4n-1} = \lim_{n\to\infty} \frac{1}{2-1/2n} = \frac{1}{2},$$

while

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{(2n+1)\cos((2n+1)\pi)}{4n+1} = \lim_{n\to\infty} -\frac{1+1/2n}{2+1/2n} = -\frac{1}{2}.$$

Since $\lim_{n\to\infty} a_{2n} \neq \lim_{n\to\infty} a_{2n+1}$, $\lim_{n\to\infty} a_n$ does not exist.

(11) The sequence $\{a_n\}$ is defined by $a_1 = 2$, and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right),$$

for $n \geq 1$. Assuming that $\{a_n\}$ converges, find its limit.

Solution: Let $a = \lim_{n\to\infty} a_n$. Since $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, we have

$$\begin{aligned} a &= \frac{1}{2}\left(a + \frac{2}{a}\right) \\ 2a^2 &= a^2 + 2 \\ a^2 &= 2. \end{aligned}$$

So $a = \sqrt{2}$ or $a = -\sqrt{2}$, where the latter is rejected since $a_n \geq 0$ (rigorous proof by mathematical induction). Therefore, $\lim_{n\to\infty} a_n = a = \sqrt{2}$.

(12) Determine whether the sequence is divergent or convergent. If it is convergent, evaluate its limit.

$$\lim_{n\to\infty} (-1)^n \sin(8/n)$$

Solution: Note that, for $n \geq 1$,

$$-|\sin(8/n)| \leq (-1)^n \sin(8/n) \leq |\sin(8/n)|.$$

Moreover, $\lim_{n\to\infty} |\sin(8/n)| = |\sin(0)| = 0$, and similarly $\lim_{n\to\infty} -|\sin(8/n)| = 0$. Therefore $\lim_{n\to\infty} (-1)^n \sin(8/n) = 0$.

In fact for $N = \left\lfloor \frac{8}{\pi/2} \right\rfloor + 1$, the tail terms $n \geq N$ satisfy

$$-8/n \leq (-1)^n \sin(8/n) \leq 8/n,$$

this is because when $n \geq N$, we have $0 < 8/n < \pi/2$ and for $0 < x < \pi/2$, the inequality $\sin(x) < x$ holds. By squeeze theorem, $\lim_{n\to\infty} (-1)^n \sin(8/n) = \lim_{n\to\infty} 8/n = 0$.

(13) Consider the sequence $a_n = \left\{ \frac{7n+1}{7n} - \frac{7n}{7n+1} \right\}$. Graph this sequence and use your graph to help you answer the following questions.

Part 1: Is the sequence bounded?

(a) Is the sequence a_n bounded above by a number?

(b) Is the sequence a_n bounded below by a number?

(c) Select all that apply: The sequence a_n is

A. bounded.
B. bounded below.
C. bounded above.
D. unbounded.

Part 2: Is the sequence monotonic?

The sequence a_n is

A. decreasing.
B. alternating
C. increasing.
D. none of the above

Part 3: Does the sequence converge?

(a) The sequence a_n is

- convergent
- divergent

(b) The limit of the sequence a_n is

Part 4: Conceptual follow up questions

(a) Select all that apply: The sequence $\left\{ (-1)^n \frac{10n^2+1}{n^2+n} \right\}$ is

A. monotonic
B. divergent
C. convergent
D. not monotonic
E. unbounded
F. bounded

(b) Select all that apply: The sequence $\left\{ \frac{10n^3+1}{n^2+n} \right\}$ is

A. unbounded
B. not monotonic
C. divergent
D. monotonic
E. convergent
F. bounded

(c) If a sequence is bounded, it ....

- must
- may or may not
- cannot

converge.

(d) If a sequence is monotonic, it ....

- must

- may or may not
- cannot

converge.

(e) If a sequence is bounded and monotonic, it ....
- must
- may or may not
- cannot

converge.

Solution: Part 1:

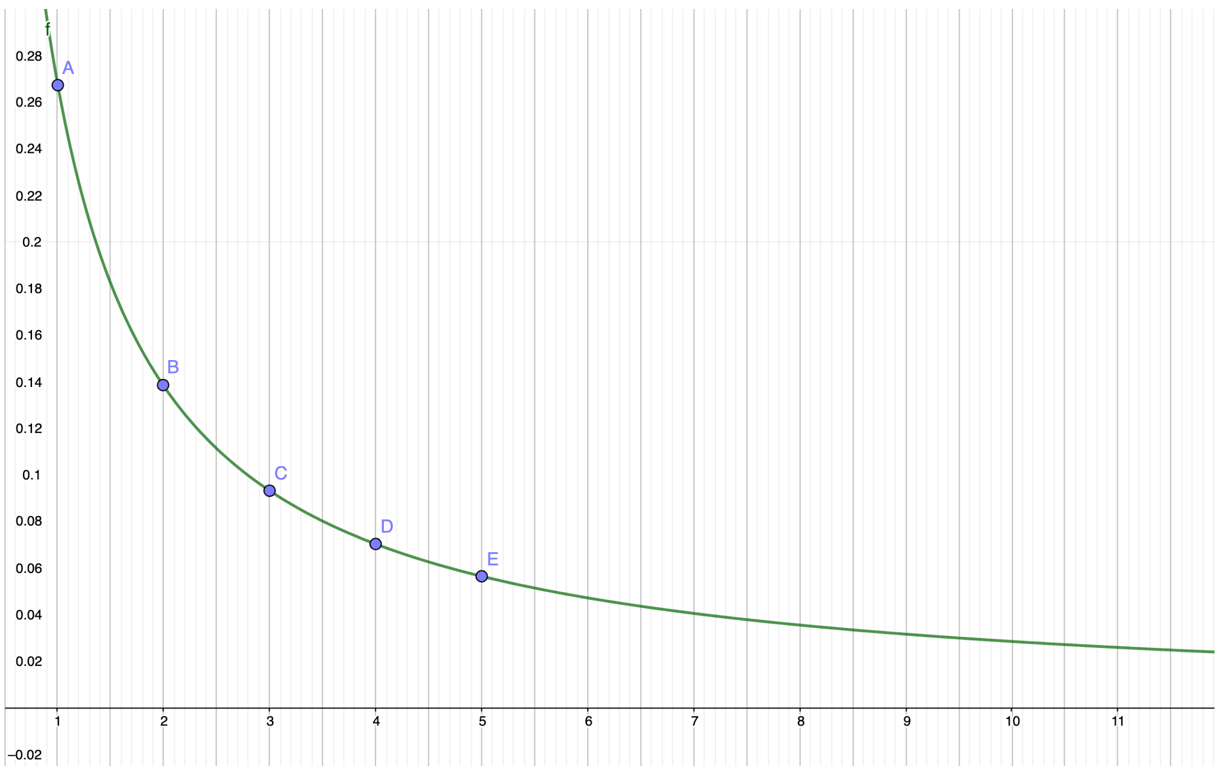
(a) Yes, the sequence is bounded above by $\frac{2}{7}$. Explanation:

$$a_n = \frac{7n+1}{7n} - \frac{7n}{7n+1} = \frac{(7n+1)^2-(7n)^2}{7n(7n+1)} = \frac{(14n+1)}{7n(7n+1)} < \frac{(14n+2)}{7n(7n+1)} = \frac{2}{7n} \leq \frac{2}{7}.$$

(b) Yes, the sequence is bounded below by 0. Explanation:

$$a_n = \frac{7n+1}{7n} - \frac{7n}{7n+1} > \frac{7n}{7n} - \frac{7n}{7n+1} > 0.$$

(c) The sequence is bounded, bounded below and bounded above (i.e A, B and C are the correct answers).



Part 2:

The sequence a_n is monotonic decreasing A (or monotonically decreasing). Explanation:

Compute the terms of this sequence to get

$$a_1 = \frac{15}{56} \approx 0.27, a_2 = \frac{29}{210} \approx 0.14, a_3 = \frac{43}{462} \approx 0.09, \ldots$$

From this we can see that the sequence is monotonically decreasing.

For the rigorous proof, note that

$$a_n = \frac{(14n+1)}{7n(7n+1)}$$

and then compute $a_{n+1} - a_n$ to find that $a_{n+1} - a_n < 0$.

Part 3:

(a) The sequence a_n is convergent because it's bounded and monotonic.

(b) The limit of the sequence a_n is 0. Proof

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{7n+1}{7n} - \frac{7n}{7n+1}\right) = \lim_{n\to\infty}\frac{7n+1}{7n} - \lim_{n\to\infty}\frac{7n}{7n+1} = 1-1=0.$$

Part 4:

(a) For n is even the sequence becomes $\left\{\frac{10n^2+1}{n^2+n}\right\}$ and $\frac{10n^2+1}{n^2+n} \leq \frac{10n^2+1}{n^2+1} < \frac{10n^2+10}{n^2+1} \leq 10$.

For n is odd the sequence becomes $\left\{-\frac{10n^2+1}{n^2+n}\right\}$ and $-\frac{10n^2+1}{n^2+n} \geq -\frac{10n^2+1}{n^2+1} > -\frac{10n^2+10}{n^2+1} \geq -10$. Thus the sequence is bounded above by 10 and bounded below by -10.

Therefore, the sequence is bounded but not monotonic because it changes sign.

For even $n = 2k$, $\frac{10n^2+1}{n^2+n} = \frac{10+1/n^2}{1+1/n}$, we have

$$\lim_{k\to\infty} a_{2k} = \lim_{n\to\infty}\frac{10+1/n^2}{1+1/n} = \frac{\lim_{n\to\infty} 10+1/n^2}{\lim_{n\to\infty} 1+1/n} = 10,$$

while for odd $n = 2k-1$, $-\frac{10n^2+1}{n^2+n} = -\frac{10+1/n^2}{1+1/n}$, we have

$$\lim_{k\to\infty} a_{2k-1} = \lim_{n\to\infty} -\frac{10+1/n^2}{1+1/n} = -\frac{\lim_{n\to\infty} 10+1/n^2}{\lim_{n\to\infty} 1+1/n} = -10,$$

The limits of even subsequence and odd subsequence do not match, therefore the sequence is divergent.

So the correct answers are B, D, and F.

(b) We denote the sequence by $a_n = \frac{10n^3+1}{n^2+n}$. Then for arbitrary n, we have

$$\begin{aligned} a_{n+1} - a_n &= \frac{10(n+1)^3+1}{(n+1)^2+(n+1)} - \frac{10n^3+1}{n^2+n} \\ &= \frac{10(n+1)^3+1}{(n+2)(n+1)} - \frac{10n^3+1}{(n+1)n} \\ &= \frac{[10(n+1)^3+1]n - (10n^3+1)(n+2)}{(n+2)(n+1)n} \\ &= \frac{10n^3+30n^2+10n-2}{(n+2)(n+1)n} \end{aligned}$$

The numerator $10n^3 + 30n^2 + 10n - 2 > 10n - 2 \geq 8 > 0$ for $n \geq 1$, so $a_{n+1} - a_n > 0$ for arbitrary $n \geq 1$, $n \in \mathbb{N}$, hence the sequence is monotonic increasing.
Note that the following inequality holds for $n \geq 1$:

$$\frac{10n^3+1}{n^2+n} > \frac{10n^3}{n^2+n} \geq \frac{10n^3}{n^2+n^2} = 5n$$

so the sequence is unbounded, hence it's divergent.
So the correct answers are A, C, D.

(c) If a sequence is bounded, it may or may not converge.
A bounded sequence may jump up and down indefinitely. Part 4 (a) is an example. The sequence $\left\{(-1)^n \frac{10n^2+1}{n^2+n}\right\}$ is bounded but not monotonic and not convergent.

(d) If a sequence is monotonic, it may or may not converge.
A sequence may monotonically tend to $+\infty$ or $-\infty$. Part 4 (b) is an example. The sequence $\left\{\frac{10n^3+1}{n^2+n}\right\}$ is monotonically increasing but unbounded, hence it is not convergent.

(e) If a sequence is bounded and monotonic, it must converge. [This is the monotonic convergence theorem.]

(14) Let $a_n = \frac{n+2}{n+5}$. Find the smallest number M such that:
(a) $|a_n - 1| \leq 0.001$ for $n \geq M$
(b) $|a_n - 1| \leq 0.00001$ for $n \geq M$
(c) Now use the limit definition to prove that $\lim_{n\to\infty} a_n = 1$. That is, find the smallest value of M (in terms of t) such that $|a_n - 1| < t$ for all $n > M$. (Note that we are using t instead of ϵ in the definition in order to allow you to enter your answer more easily).

Solutions:

(a) We have

$$|a_n - 1| = \left|\frac{n+2}{n+5} - 1\right| = \left|\frac{n+2-(n+5)}{n+5}\right| = \left|\frac{-3}{n+5}\right| = \frac{3}{n+5}$$

Therefore $|a_n - 1| \leq 0.001$ provided $\frac{3}{n+5} \leq 0.001$, that is, $n \geq 2995$. It follows that we can take $M = 2995$.

(b) By part (a), $|a_n - 1| \leq 0.00001$ provided $\frac{3}{n+5} \leq 0.00001$, that is, $n \geq 299995$. It follows that we can take $M = 299995$.

(c) Using part (a), we know that

$$|a_n - 1| = \frac{3}{n+5} < t,$$

provided $n > \frac{3}{t} - 5$. Thus to complete the proof, let $t > 0$ and take $M = \frac{3}{t} - 5$. Then, for $n > M$, we have

$$|a_n - 1| = \frac{3}{n+5} < \frac{3}{M+5} = t.$$

(15) Consider the sequence $a_n = \left\{ \dfrac{(-1)^n \cdot 6n}{n+1} \right\}$. Graph this sequence and use your graph to help you answer the following questions.

Part 1: Is the sequence bounded?

(a) Is the sequence a_n bounded above by a function? If it is, enter the function of the variable n that provides the "best" and "most obvious" upper bound.
(b) What is the limit of the function from part (a) as $n \to \infty$?
(c) Is the sequence $\{a_n\}$ bounded below by a function? If it is, enter the function of the variable n that provides the "best" and "most obvious" lower bound.
(d) What is the limit of the function from part (c) as $n \to \infty$?
(e) Is the sequence $\{a_n\}$ bounded above by a number?
(f) Is the sequence $\{a_n\}$ bounded below by a number?
(g) Select all that apply: The sequence $\{a_n\}$ is
A. bounded below.
B. bounded above.
C. bounded.
D. unbounded.

Part 2: Is the sequence monotonic?

The sequence a_n is
A. decreasing.
B. alternating
C. increasing.
D. none of the above

Part 3: Does the sequence converge?

(a) The sequence a_n is
- convergent
- divergent

(b) The limit of the sequence a_n is

Part 4: Conceptual follow up questions

(a) When you first look at the sequence $\left\{ \dfrac{(-1)^n \cdot 6n}{n+1} \right\}$, you expect it to
A. converge to both -6 and 6 because the odd index terms tend to -6 as $n \to \infty$, while the even index terms tend to 6 as $n \to \infty$.
B. diverge because the odd index terms tend to -6 as $n \to \infty$, while the even index terms tend to 6 as $n \to \infty$, so there is not one single value for the limit of the sequence.
C. diverge because alternating sequences always diverge.

(b) When you first look at the sequence $\left\{ \dfrac{(-1)^n \cdot 6}{n+1} \right\}$, you expect it to
A. converge to 0 because $-\frac{6}{n+1} \le \frac{(-1)^n \cdot 6}{n+1} \le \frac{6}{n+1}$ and both $\frac{-6}{n+1}$ and $\frac{6}{n+1}$ converge to 0 as $n \to \infty$.
B. diverge because the odd index terms tend to -6 as $n \to \infty$, while the even index terms tend to 6 as $n \to \infty$, so there is not one single value for the limit of the sequence.
C. diverge because alternating sequences always diverge.

(c) If a sequence is alternating, it
- must
- may or may not

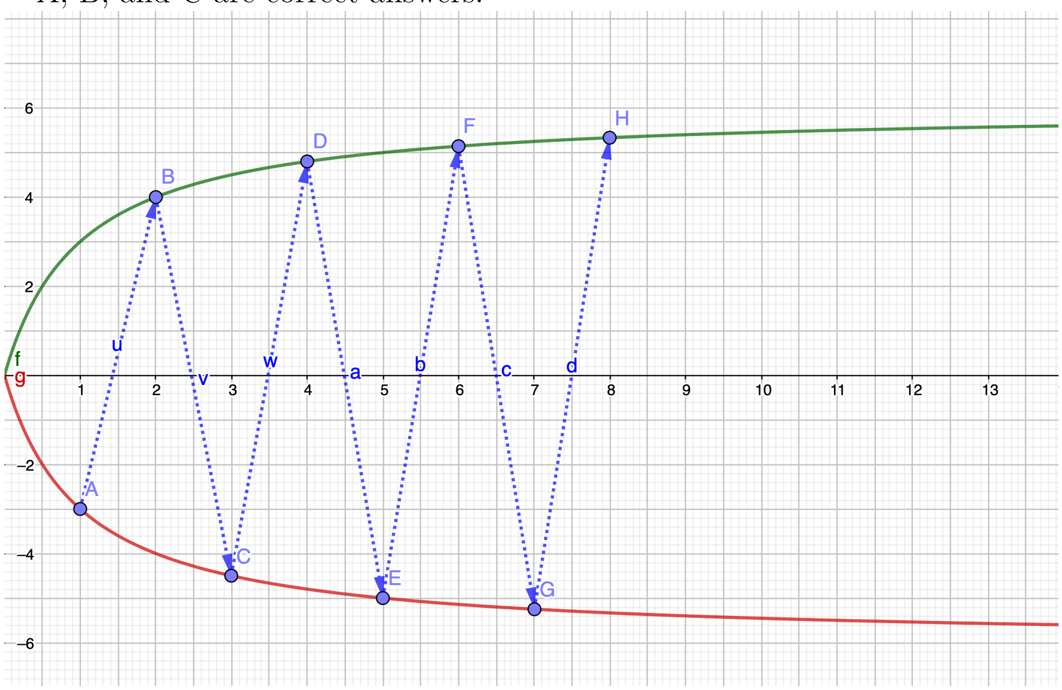
- cannot

converge.

Solutions:

Part 1:

(a) Just take the absolute value of a_n and get
$$a_n = \frac{(-1)^n \cdot 6n}{n+1} \le \left|\frac{(-1)^n \cdot 6n}{n+1}\right| = \frac{6n}{n+1}.$$
(b) Note that
$$\lim_{n\to\infty} \frac{6n}{n+1} = 6.$$
(c) Just take the minus absolute value of a_n
$$a_n = \frac{(-1)^n \cdot 6n}{n+1} \ge -\left|\frac{(-1)^n \cdot 6n}{n+1}\right| = -\frac{6n}{n+1}.$$
(d) Note that
$$\lim_{n\to\infty} -\frac{6n}{n+1} = -6.$$
(e) By (a),
$$a_n \le \frac{6n}{n+1} < \frac{6n+6}{n+1} = 6.$$
(f) By (c),
$$a_n \ge -\frac{6n}{n+1} > -\frac{6n+6}{n+1} = -6.$$
(g) By (e) and (f), we know that a_n is bounded above by 6 and below by -6, so A, B, and C are correct answers.



Part 2: Since $(-1)^n$ is in the numerator, the sequence is alternating B.

Part 3:

(a) By Part 1 (b), (d), and Part 2, the sequence is alternating between two different values, so it is divergent.

(b) The limit does not exist (DNE).

Part 4:

(a) By Part 3, the correct answer is B: it diverges because the odd index terms tend to -6 as $n \to \infty$, while the even index terms tend to 6 as $n \to \infty$, so there is not one single value for the limit of the sequence.

(b) For $\left\{ \dfrac{(-1)^n \cdot 6}{n+1} \right\}$, it converges to 0 because $-\frac{6}{n+1} \leq \frac{(-1)^n \cdot 6}{n+1} \leq \frac{6}{n+1}$ and both $\frac{-6}{n+1}$ and $\frac{6}{n+1}$ converge to 0 as $n \to \infty$. The correct answer is A.

(c) By (a) and (b), an alternating sequence may or may not converge.

(16) Find the domain, x-intercept(s), y-intercept(s), and symmetry of the function

$$f(x) = 4 - x^2.$$

Solutions:

(a) The domain of f is $(-\infty, \infty)$.

(b) We solve the equation

$$\begin{aligned} f(x) &= 0 \\ 4 - x^2 &= 0 \\ x &= \pm 2 \end{aligned}$$

Hence, the x-intercepts of f are $(-2, 0)$ and $(2, 0)$.

(c) Taking $x = 0$, we have $f(x) = 4$.
Hence, the y-intercept of f is $(0, 4)$.

(d) Since $f(0) \neq 0$, f is not odd.
Since $f(-x) = 4 - (-x)^2 = 4 - x^2 = f(x)$, f is even.

(17) The domain of the function $f(x) = \dfrac{\sqrt{4 - x^2}}{\sqrt{1 - x^2}}$ is the interval

Solution: The numerator is defined on $[-2, 2]$ and the denominator is defined and non-zero on $(-1, 1)$, which is contained in the domain of the numerator. Hence the domain of the function is $(-1, 1)$.