THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2022-2023 Term 1
Suggested Solutions of WeBWork Coursework 10

1. (1 point) **Integration by Parts:** This is the most important integration technique we've discussed in this class. It has a wide range of applications beyond increasing our list of integration rules.

$\int z^3 \ln z \mathrm{d}z =$ ________________.

$\int e^t \cos t \mathrm{d}t =$ ________________.

$\int_0^{2\pi} \sin(x)\sin(x+1)\mathrm{d}x =$________________.

Solution:

$$
\begin{aligned}
&\int z^3 \ln z \, dz \\
&= \int \ln z \, d\frac{1}{4}z^4 = \frac{1}{4}\int \ln z \, dz^4 \\
&= \frac{1}{4}[(\ln z)z^4 - \int z^4 \, d\ln z] \\
&= \frac{1}{4}[(\ln z)z^4 - \int z^3 \, dz] \\
&= \frac{1}{4}[(\ln z)z^4 - \frac{1}{4}z^4] + C = \frac{(4\ln z - 1)z^4}{16} + C
\end{aligned}
$$

$$
\begin{aligned}
&\int e^t \cos t \, dt \\
&= \int e^t \, d\sin t \\
&= e^t \sin t - \int e^t \sin t \, dt \\
&= e^t \sin t + \int e^t \, d\cos t \\
&= e^t \sin t + e^t \cos t - \int e^t \cos t \, dt
\end{aligned}
$$

Then,

$$
\begin{aligned}
2\int e^t \cos t \, dt &= e^t(\sin t + \cos t) + C \\
\int e^t \cos t \, dt &= \frac{1}{2}e^t(\sin t + \cos t) + C
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{2\pi} \sin(x)\sin(x+1)\,dx \\
&= \int_0^{2\pi} \frac{\cos(-1)-\cos(2x+1)}{2}\,dx \\
&= \frac{cos(-1)}{2} x\Big|_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \cos(2x+1)\,dx \\
&= \pi\cos 1 - \frac{1}{4}\sin(2x+1)\Big|_0^{2\pi} \\
&= \pi\cos 1 - \frac{1}{4}\sin(4\pi+1) + \frac{1}{4}\sin(1) \\
&= \pi\cos 1 \approx 1.69740975483297
\end{aligned}
$$

---

2. (1 point)
Evaluate the integral

$$\int_0^{\pi/3} \frac{7\sin(x)+7\sin(x)\tan^2(x)}{\sec^2(x)}\,dx$$

Integral = ____________

Solution:

$$
\begin{aligned}
&\int_0^{\pi/3} \frac{7\sin(x)+7\sin(x)\tan^2(x)}{\sec^2(x)}\,dx \\
&= \int_0^{\pi/3} \frac{7\sin(x)\sec^2(x)}{\sec^2(x)}\,dx \\
&= \int_0^{\pi/3} 7\sin(x)\,dx \\
&= -7\cos(x)\Big|_0^{\pi/3} \\
&= \frac{7}{2}
\end{aligned}
$$

---

3. (1 point)
Evaluate the integral

$$\int_0^4 \left|\sqrt{x+2}-x\right|\,dx$$

____

Solution:

$$
\begin{aligned}
&\int_0^4 |\sqrt{x+2} - x|\, dx \\
&= \int_0^2 (\sqrt{x+2} - x)\, dx - \int_2^4 (\sqrt{x+2} - x)\, dx \\
&= \left(\frac{2}{3}(x+2)^{3/2} - \frac{1}{2}x^2\right)\Big|_0^2 - \left(\frac{2}{3}(x+2)^{3/2} - \frac{1}{2}x^2\right)\Big|_2^4 \\
&= \left[\left(\frac{16}{3} - 2\right) - \frac{4}{3}\sqrt{2}\right] - \left[\left(4\sqrt{6} - 8\right) - \left(\frac{16}{3} - 2\right)\right] \\
&= \frac{44}{3} - 4\sqrt{6} - \frac{4}{3}\sqrt{2}
\end{aligned}
$$

---

4. (1 point)
Evaluate the integral

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} -1t^3 \cos(t^2)\, dt$$

Solution: Evaluate the indefinite integral

$$\int -1t^3 \cos(t^2)\, dt = -\int t^3 \cos(t^2)\, dt$$

Let $u = t^2$, $du = 2tdt$,

$$
\begin{aligned}
-\int t^3 \cos(t^2)\, dt &= -\int \frac{1}{2} u \cos(u)\, du \\
&= -\frac{1}{2}\int u\, d\sin(u) \\
&= -\frac{1}{2}(u \sin u - \int \sin u\, du) \\
&= -\frac{1}{2}(u \sin u + \cos u)
\end{aligned}
$$

Substitute back,

$$-\int t^3 \cos(t^2)\, dt = -\frac{t^2 \sin(t^2) + \cos(t^2)}{2}$$

Then, the definite integral

$$
\begin{aligned}
&\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} -1t^3 \cos(t^2)\, dt \\
&= -\frac{t^2 \sin(t^2) + \cos(t^2)}{2}\Big|_{\sqrt{\pi/2}}^{\sqrt{\pi}} \\
&= -\frac{\pi \sin(\pi) + \cos(\pi)}{2} + \frac{\frac{\pi}{2}\sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2})}{2} = \frac{1}{2} + \frac{\pi}{4}
\end{aligned}
$$

---

5. (1 point)

Evaluate the definite integral:

$\int_{-4}^{3} (x - 4|x|)\, dx =$ ____________________

Solution:

$$
\begin{aligned}
&\int_{-4}^{3} (x - 4|x|)\, dx \\
&= \int_{-4}^{0} 5x\, dx - \int_{0}^{3} 3x\, dx \\
&= \frac{5}{2}x^2\Big|_{-4}^{0} - \frac{3}{2}x^2\Big|_{0}^{3} \\
&= -40 - \frac{27}{2} \\
&= -\frac{107}{2}
\end{aligned}
$$

---

6. (1 point) Find the definite integrals:

(a) $\int_0^1 \frac{x^4+4x+8}{x^2+2x+2} dx =$ ____________________

(b) $\int_0^1 \frac{x^4+5}{x^2+2x+2} dx =$ ____________________

Solution:

(a) Evaluate the indefinite integral

$$
\begin{aligned}
&\int \frac{x^4 + 4x + 8}{x^2 + 2x + 2}\, dx \\
&= \int x^2 - 2x + 2 + \frac{4x + 4}{x^2 + 2x + 2}\, dx \\
&= \int x^2\, dx - \int 2x\, dx + \int 2\, dx + \int \frac{4x + 4}{x^2 + 2x + 2}\, dx \\
&= \frac{1}{3}x^3 - x^2 + 2x + 2\ln(|x^2 + 2x + 2|)
\end{aligned}
$$

Then, the definite integral

$$
\begin{aligned}
&\int_0^1 \frac{x^4 + 4x + 8}{x^2 + 2x + 2}\, dx \\
&= \frac{1}{3}x^3\Big|_0^1 - x^2\Big|_0^1 + 2x\Big|_0^1 + 2\ln(|x^2 + 2x + 2|)\Big|_0^1 \\
&= \frac{1}{3} - 1 + 2 + 2\ln(\frac{5}{2}) = \frac{4}{3} + 2\ln(\frac{5}{2}) \\
&\approx 3.16591479708164
\end{aligned}
$$

(b) Evaluate the indefinite integral

$$\begin{aligned}
&\int \frac{x^4+5}{x^2+2x+2}\,dx \\
&= \int x^2 - 2x + 2 + \frac{1}{x^2+2x+2}\,dx \\
&= \int x^2\,dx - \int 2x\,dx + \int 2\,dx + \int \frac{1}{x^2+2x+2}\,dx \\
&= \int x^2\,dx - \int 2x\,dx + \int 2\,dx + \int \frac{1}{(x+1)^2+1}\,dx \\
&= \frac{1}{3}x^3 - x^2 + 2x + \arctan(x+1)
\end{aligned}$$

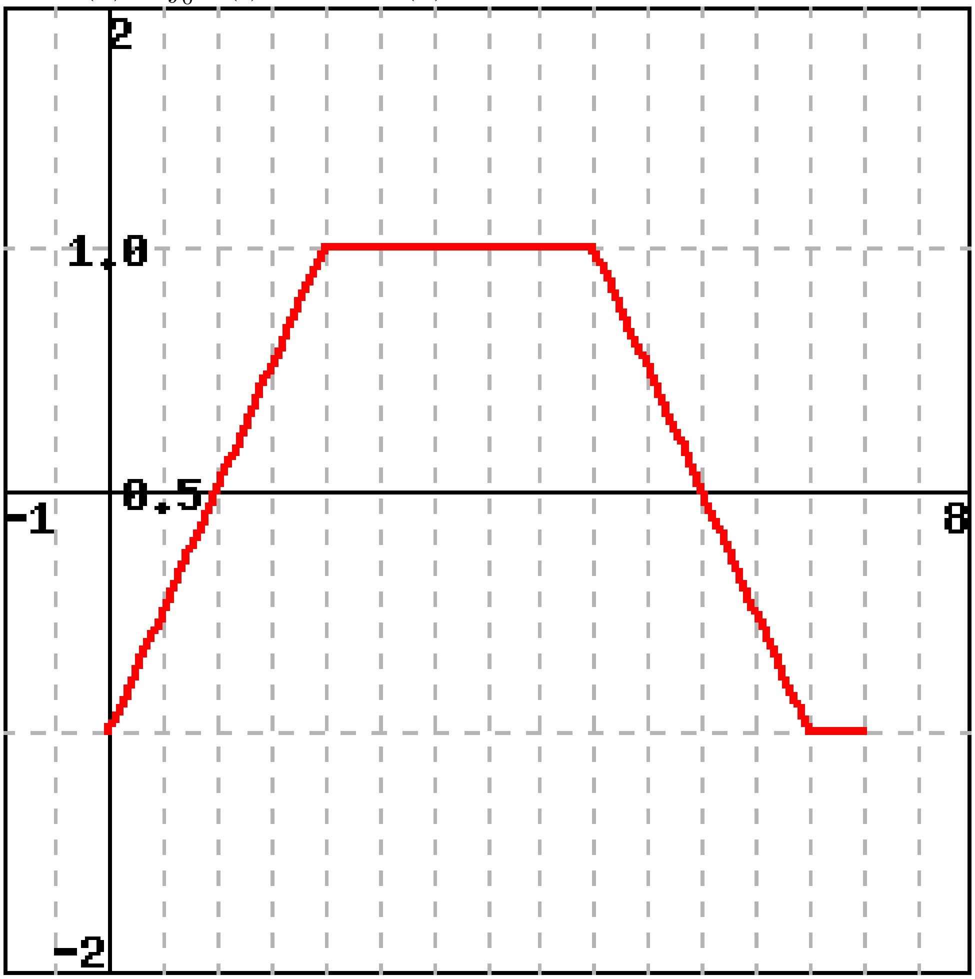
Then, the definite integral

$$\begin{aligned}
&\int_0^1 \frac{x^4+5}{x^2+2x+2}\,dx \\
&= \frac{1}{3}x^3\Big|_0^1 - x^2\Big|_0^1 + 2x\Big|_0^1 + \arctan(x+1)\Big|_0^1 \\
&= \frac{4}{3} + \arctan(2) - \frac{\pi}{4} \\
&\approx 1.65508388772998
\end{aligned}$$

---

7. (1 point)

Let $A(x) = \int_0^x f(t)\,dt$, with $f(x)$ as in figure.



$A(x)$ has a local minimum on $(0,6)$ at $x =$ __________

$A(x)$ has a local maximum on $(0,6)$ at $x =$ __________

Solution:

The minimum values of $A(x)$ on $(0,6)$ occur where $A'(x) = f(x)$ goes from negative to positive. This occurs at one place, where $x = 1$. The maximum values of $A(x)$ on $(0,6)$ occur where $A'(x) = f(x)$ goes from positive to negative. This occurs at one place, where $x = 5.5$.

---

8. (1 point)
$G(x) = \int_1^x \tan t\,dt$

Find $G(1) =$ __________

Find $G'(\frac{\pi}{5}) =$ __________

Solution:

By definition $G(1) = \int_1^1 \tan t\,dt = 0$.
By the FTC, part II, $G'(x) = \tan x$, so $G'(\frac{\pi}{5}) = \tan\frac{\pi}{5}$.

9. (1 point) Given

$$f(x) = \int_0^x \frac{t^2 - 16}{1 + \cos^2(t)} dt$$

At what value of x does the local max of $f(x)$ occur?

$x =$ ________

Solution:

First, note that we don't need to do any computation to compute the first derivative, which we will use to check for local maxima and minima. By applying the Fundamental Theorem of Calculus, we see that:

$$f'(x) = \frac{x^2 - 16}{1 + \cos^2(x)}$$

Now, we can use this derivative to find the critical points of the function. We set this to zero and solve for x to get:

$$\frac{x^2 - 16}{1 + \cos^2(x)} = 0$$

$$x^2 - 16 = 0$$

$$(x + 4)(x - 4) = 0$$

$$x = 4 \text{ or } x = -4$$

Checking on either side of these two points shows that -4 is the local maximum for which we are looking.

---

10. (1 point)

Let

$$f(x) = \begin{cases} 0 & \text{if } x < -4 \\ 5 & \text{if } -4 \le x < -1 \\ -2 & \text{if } -1 \le x < 5 \\ 0 & \text{if } x \ge 5 \end{cases}$$

and

$$g(x) = \int_{-4}^x f(t) dt$$

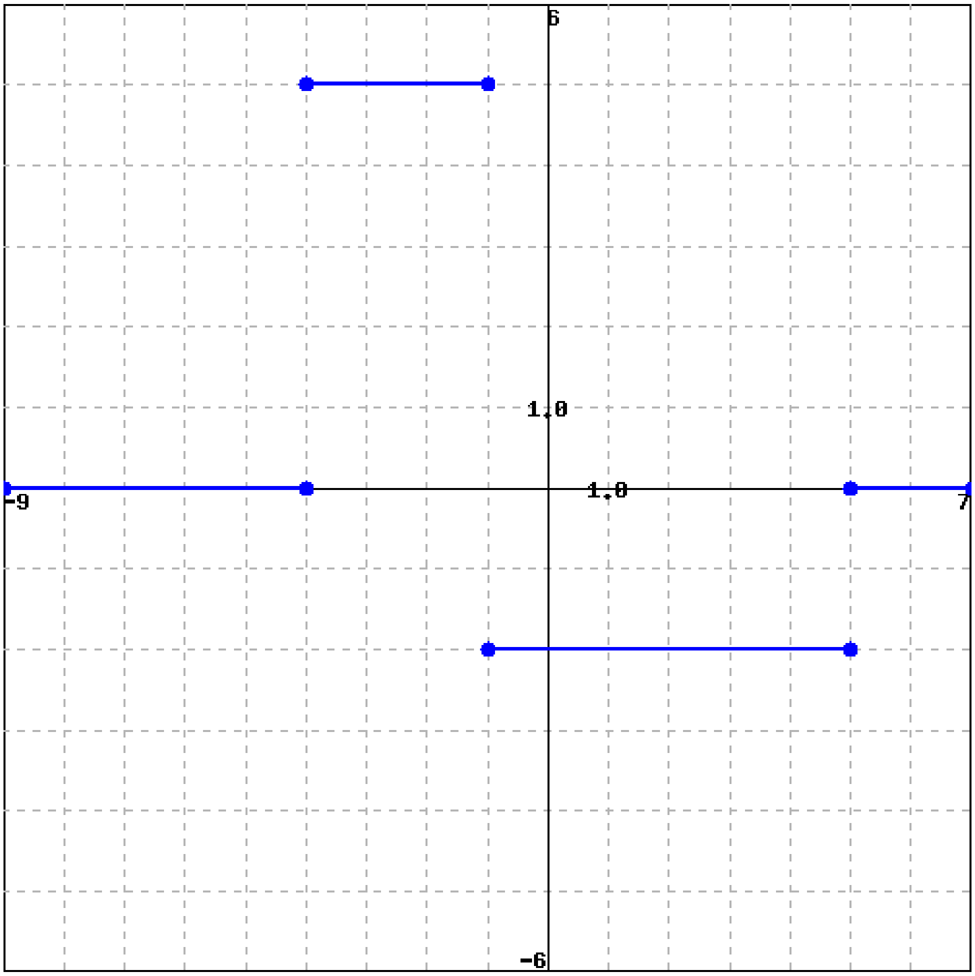
Determine the value of each of the following:
(a) $g(-6) =$ ___
(b) $g(-3) =$ ___
(c) $g(0) =$ ___
(d) $g(6) =$ ___
(e) The absolute maximum of $g(x)$ occurs when $x =$ ___ and is the value ___

It may be helpful to make a graph of $f(x)$ when answering these questions.

Solution:
As the problem statement suggests, the best method to use in solving this problem is to

graph the function $f(x)$. Then, we can use the fact that the integration of a curve on the interval [a,b] can be interpreted as the area underneath that curve between the lines $x = a$ and $x = b$.

The graph of $f(x)$ is shown below.



(a) $g(-6)$ will be the area under the graph of $f(x)$ for $-6 \leq x \leq -4$. As f is always 0 on that interval in the graph, the area underneath it is clearly 0, making $g(-6) = 0$.

(b) In this case, we want the area underneath the graph of $f(x)$ on the interval $-4 \leq x \leq -3$. On this interval, f is always 5. Therefore, the value of $g(-3)$ is 5.

(c) To compute $g(0)$, note that f takes on two different values on the interval [-4,0]. Between -4 and -1, f has the value 5. Between -1 and 0, f has the value -2. Therefore, the value of $g(0)$ is $5(-1 + 4) - 2(0 + 1) = 13$.

(d) In computing $g(6)$, we note that again, f takes on different values on the interval [-4,6]. Between -4 and -1, the value is 5. Between -1 and 5 the value is -2. Finally, between 5 and 6, the value is 0. Hence, the value of $g(6)$ is $5(-1+4) - 2(5+1) + 0(6-5) = 3$.

(e) The maximum value of g occurs at the x for which f has the largest area between -4 and x. Note that between -1 and 5, f actually has a negative value, which would take away from the value of g. Therefore, the maximum value will take place at -1. That value will be the total area underneath the graph between -4 and -1, which is 15.

---

11. (1 point)

Use the Fundamental Theorem of Calculus to evaluate (if it exists)

$$\int_{-\pi}^{\pi} f(x)\,dx,$$

where

$$f(x) = \begin{cases} 6x & \text{if } -\pi \le x \le 0 \\ -8\sin(x) & \text{if } 0 < x \le \pi \end{cases}$$

If the integral does not exist, type "DNE" as your answer.

____

Solution:

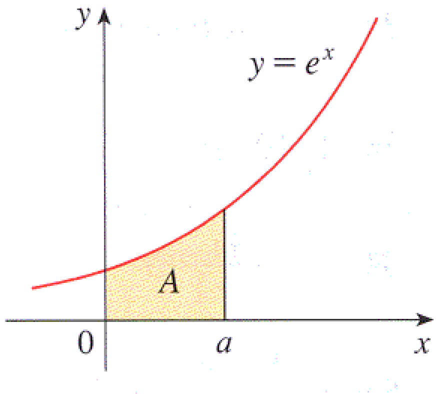
$$\begin{aligned} &\int_{-\pi}^{\pi} f(x)\,dx \\ &= \int_{-\pi}^{0} 6x\,dx + \int_{0}^{\pi} -8\sin(x)\,dx \\ &= 3x^2\Big|_{-\pi}^{0} + 8\cos(x)\Big|_{0}^{\pi} \\ &= -3\pi^2 - 8 - 8 = -3\pi^2 - 16 \end{aligned}$$

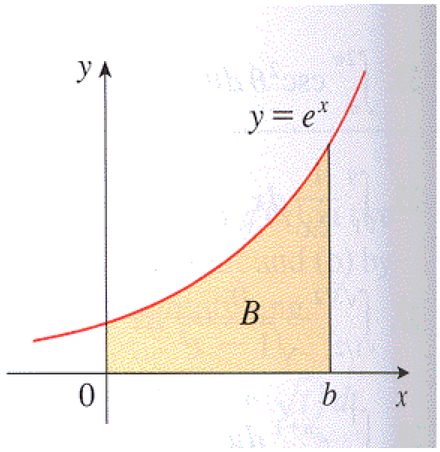
---

12. (1 point)

The area labeled B is 2 times the area labeled A. Express b in terms of a.

$b =$ ______________





Solution:

$$\int_0^b e^x dx = 2\int_0^a e^x\, dx$$
$$e^b - 1 = 2(e^a - 1)$$
$$b = \ln(2e^a - 2 + 1) = \ln(2e^a - 1)$$

---

13. (1 point)

For each of the following integrals, indicate whether integration by substitution or integration by parts is more appropriate, or if neither method is appropriate. Do not evaluate the integrals.

1. $\int x \sin x\, dx$

- A. neither
- B. substitution
- C. integration by parts

2. $\int \frac{x^4}{1+x^5}\, dx$

- A. substitution
- B. integration by parts
- C. neither

3. $\int x^4 e^{x^5}\, dx$

- A. integration by parts
- B. substitution
- C. neither

4. $\int x^4 \cos(x^5)\, dx$

- A. substitution
- B. integration by parts
- C. neither

5. $\int \frac{1}{\sqrt{8x+1}}\, dx$

- A. integration by parts
- B. neither

- C. substitution

(Note that because this is multiple choice, you will not be able to see which parts of the problem you got correct.)

Solution:

For each of these, we're looking to see if there is a good substitution (we can take w to be the argument of a function, etc., such that its derivative, $dw = w'\,dx$, appears in the integrand; or, if the function looks like one on which integration by parts is good. We see that:

1. For $\int x \sin x\, dx$, integration by parts is appropriate, because by taking $u = x$ and $v' = \sin(x)$, we end up with an integral that is easy to find.

2. For $\int \frac{x^4}{1+x^5}\, dx$, substitution is appropriate, because by taking $w = 1 + x^5$ we get an integral we can find.

3. For $\int x^4 e^{x^5}\, dx$, substitution is appropriate, because by taking $w = x^5$ we get an integral we can find.

4. For $\int x^4 \cos(x^5)\, dx$, substitution is appropriate, because by taking $w = x^5$ we get an integral we can find.

5. For $\int \frac{1}{\sqrt{8x+1}}\, dx$, substitution is appropriate, because by taking $w = 8x + 1$ we get an integral we can find.

Correct Answers:

- `C`
- `A`
- `B`
- `A`
- `C`

---

14. (1 point)

What is the correct form of the partial fraction decomposition for the following integral?

$$\int \frac{x^2+1}{(x-1)^3(x^2+11x+45)}\, dx$$

- A. There is no partial fraction decomposition yet because long division must be done first.
- B. $\displaystyle\int \left(\frac{A}{(x-1)^3} + \frac{B}{x-11} + \frac{C}{(x-11)^2} + \frac{Dx+E}{x^2+1} \right) dx$
- C. $\displaystyle\int \left(\frac{A}{(x-1)^3} + \frac{Bx+C}{x^2+11x+45} \right) dx$
- D. $\displaystyle\int \left(\frac{A}{x-1} + \frac{Bx+C}{(x-1)^2} + \frac{Dx+E}{(x-1)^3} + \frac{Fx+G}{x^2+11x+45} \right) dx$
- E. $\displaystyle\int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+11x+45} \right) dx$
- F. $\displaystyle\int \left(\frac{A}{(x-1)^3} + \frac{B}{x-11} + \frac{C}{x-45} \right) dx$
- G. There is no partial fraction decomposition because the denominator does not factor.
- H. There is no partial fraction decomposition yet because there is cancellation.

Solution:

Note that $x^2+11x+45$ is an irreducible quadratic since $b^2-4ac = (11)^2-4(45) = -59 < 0$. Since the denominator factors in the linear term $x-1$ repeated three times, and in an irreducible quadratic, the correct form of the partial fraction is:

$$\int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+11x+45} \right) dx$$

Thus the correct answer is **E.**

---

15. (1 point) Find a function f and a number a such that

$$1 + \int_a^x \frac{f(t)}{t^7} dt = 5x^{-3}$$

$f(x) =$ __________

$a =$ __________

Solution:

The best way to solve this problem is to first determine the form of the function $f(x)$ by using the first portion of the Fundamental Theorem of Calculus to find the the derivative. First, we rearrange the equation as shown below:

$$\int_a^x \frac{f(t)}{t^7} dt = 5x^{-3} - 1$$

Next, we differentiate both sides to get:

$$\frac{f(x)}{x^7} = -15x^{-4}$$

Therefore, the function desired is:

$$f(x) = -15x^3$$

Now, to determine the value of a, we need to integrate the original expression using our new information about $f(t)$. That is, we now have:

$$\begin{aligned}
1 + \int_a^x \frac{-15t^3}{t^7} dt &= 5x^{-3} \\
\int_a^x -15t^{-4} dt &= 5x^{-3} - 1 \\
5x^{-3} - 5a^{-3} &= 5x^{-3} - 1 \\
5a^{-3} &= 1 \\
a^{-3} &= \frac{1}{5} \\
\frac{1}{a^3} &= \frac{1}{5} \\
a &= \sqrt[3]{\frac{5}{1}}
\end{aligned}$$

---

16. (1 point) Compute the following limit. Use INF to denote ∞ and MINF to denote $-\infty$.

$\lim_{x\to 0} \frac{x}{\int_x^{x^2} \sqrt[3]{216-7t^3}\,dt} =$ __________

Solution:

$$\begin{aligned}
&\lim_{x\to 0} \frac{x}{\int_x^{x^2} \sqrt[3]{216-7t^3}\,dt} && \left(\frac{0}{0}\right) \\
&= \lim_{x\to 0} \frac{1}{2x\sqrt[3]{216-7x^6}-\sqrt[3]{216-7x^3}} && \text{(using L'Hopital rule)} \\
&= \frac{1}{-6} = -\frac{1}{6}
\end{aligned}$$

---

17. (1 point)

Evaluate the limit $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{3j}{n^2}$.

$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{3j}{n^2} =$ __________

Solution:

Let $S_n = \sum_{j=1}^{n} \frac{3j}{n^2} = \frac{3}{n^2}\sum_{j=1}^{n} j = \frac{3}{n^2}\left(\frac{n^2}{2}+\frac{n}{2}\right) = \frac{3}{2}+\frac{3}{2n}$.

Then $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{3j}{n^2} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{2}+\frac{3}{2n} = \frac{3}{2}$.

---

18. (1 point) The following sum

$$\sqrt{36-\left(\frac{6}{n}\right)^2}\cdot\frac{6}{n}+\sqrt{36-\left(\frac{12}{n}\right)^2}\cdot\frac{6}{n}+\ldots+\sqrt{36-\left(\frac{6n}{n}\right)^2}\cdot\frac{6}{n}$$

is a right Riemann sum with n subintervals of equal length for the definite integral

$$\int_0^b f(x)\,dx$$

where $b =$ __________
and $f(x) =$ __________

Solution:

Since

$$\sqrt{36-\left(\frac{6}{n}\right)^2}\cdot\frac{6}{n}+\sqrt{36-\left(\frac{12}{n}\right)^2}\cdot\frac{6}{n}+\ldots+\sqrt{36-\left(\frac{6n}{n}\right)^2}\cdot\frac{6}{n}$$

is a right Riemann sum with n subintervals of equal length for the definite integral

$$\int_0^b f(x)\,dx$$

and

$$f(0+\Delta x)\Delta x + f(0+2\Delta x)\Delta x + \cdots + f(0+n\Delta x)\Delta x.$$

Then we can get

$$\Delta x = \frac{6}{n}.$$

Since $\Delta x = \frac{b-0}{n}$, then $b = 6$.

By $f(0+\Delta x) = f(\frac{6}{n}) = \sqrt{36 - (\frac{6}{n})^2}$, then we get $f(x) = \sqrt{36 - x^2}$.