THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2022-2023 Term 1
Homework Assignment 2
Suggested Solutions for HW2

1. The function f is continuous at $x = 0$ and is defined for $-1 < x < 1$ by

$$f(x) = \begin{cases} \dfrac{2a}{x}(e^x - 1) & \text{if } -1 < x < 0 \\ 1 & \text{if } x = 0 \\ \dfrac{bx\cos x}{1 - \sqrt{1-x}} & \text{if } 0 < x < 1. \end{cases}$$

Determine the values of the constants a and b.

Solution

For f to be continuous at $x = 0$,

(a) $\lim_{x\to 0+} f(x) = f(0)$

$$\begin{aligned} 1 &= \lim_{x\to 0+} \frac{bx\cos x}{1-\sqrt{1-x}} \\ &= \lim_{x\to 0+} \frac{bx\cos x(1+\sqrt{1-x})}{1-(1-x)} \\ &= \lim_{x\to 0+} b\cos x(1+\sqrt{1-x}) \\ &= 2b \end{aligned}$$

So $b = \frac{1}{2}$.

(b) $\lim_{x\to 0-} f(x) = f(0)$

$$\begin{aligned} 1 &= \lim_{x\to 0-} \frac{2a}{x}(e^x - 1) \\ &= 2a \end{aligned}$$

So $a = \frac{1}{2}$.

2. Determine whether the following functions are differentiable at $x = 0$.

(a) $f(x) = \begin{cases} 1 + 3x - x^2, & \text{when } x < 0 \\ x^2 + 3x + 2, & \text{when } x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

(c) $f(x) = \sin|x|$

(d) $f(x) = x|x|$

Solution

(a) Note that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2+3x+2$$
$$= 2$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1+3x-x^2$$
$$= \lim_{x\to 0^-} 1 \neq 2$$

Hence, f is not continuous at $x=0$, thus not differentiable at $x=0$.

(b)

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}}}{x}$$
$$= \lim_{y\to\infty} ye^{-y^2} \quad (\text{Let } y = \frac{1}{x})$$
$$= \lim_{y\to\infty} \frac{y}{e^{y^2}}$$
$$= \lim_{y\to\infty} \frac{1}{2ye^{y^2}} \quad (\text{L'Hopital})$$
$$= 0$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{e^{-\frac{1}{x^2}}}{x}$$
$$= \lim_{y\to -\infty} ye^{-y^2} \quad (\text{Let } y = \frac{1}{x})$$
$$= \lim_{y\to -\infty} \frac{y}{e^{y^2}}$$
$$= \lim_{y\to -\infty} \frac{1}{2ye^{y^2}} \quad (\text{L'Hopital})$$
$$= 0$$

Hence, f is differentiable at $x=0$.

(c)

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\sin|x|-0}{x}$$
$$= \lim_{x\to 0^+} \frac{\sin x}{x}$$
$$= 1$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{\sin|x|-0}{x}$$
$$= \lim_{x\to 0^-} \frac{-\sin x}{x}$$
$$= -1 \neq 1$$

Hence, f is not differentiable at $x = 0$.
(d)

$$\begin{aligned}\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^+} \frac{x|x| - 0}{x} \\ &= \lim_{x \to 0^+} \frac{x^2}{x} \\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^-} \frac{x|x| - 0}{x} \\ &= \lim_{x \to 0^-} \frac{-x^2}{x} \\ &= 0\end{aligned}$$

Hence, f is differentiable at $x = 0$.

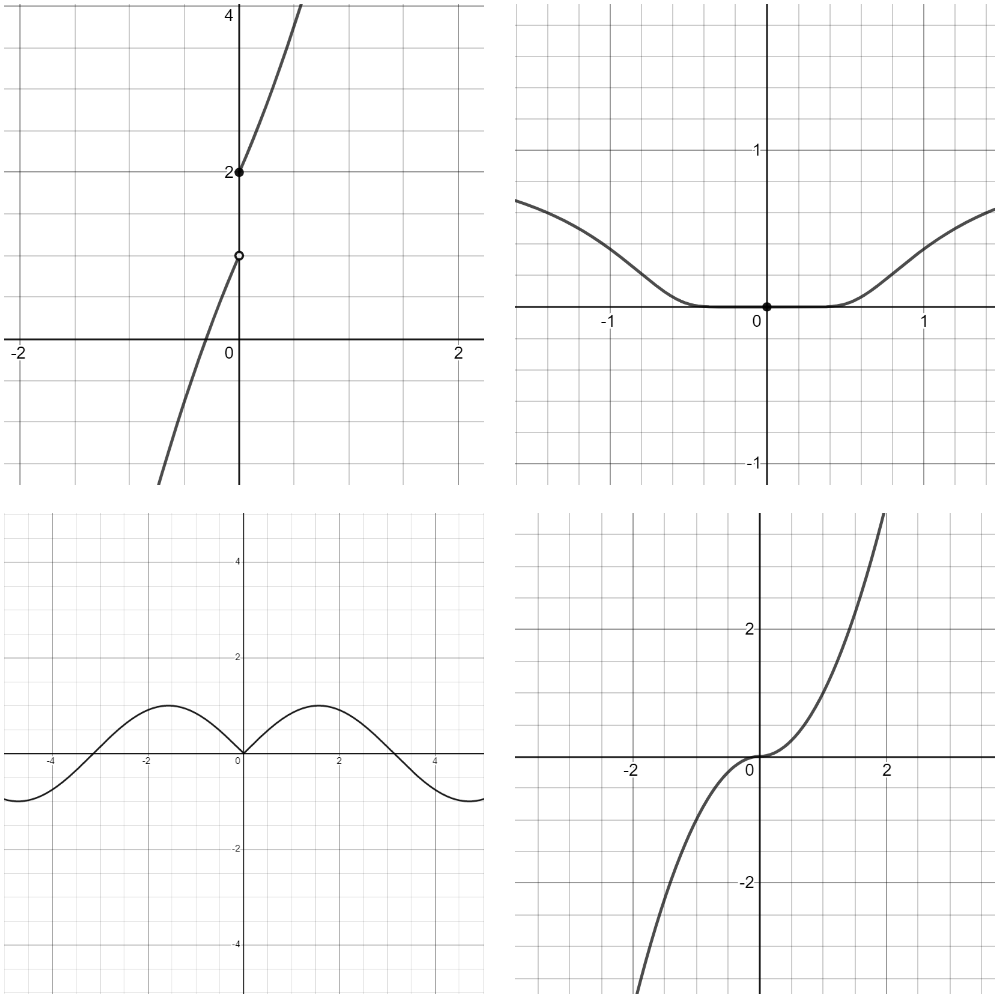


Figure 1: Graph of Q2

3. Let $f(x) = |x|^3$.

   (a) Find $f'(x)$ for $x \neq 0$.

   (b) Show that $f(x)$ is differentiable at $x = 0$.

   (c) Determine whether $f'(x)$ is differentiable at $x = 0$.

   Solution (a)

$$f'(x) = \begin{cases} 3x^2, & \text{when } x > 0; \\ -3x^2, & \text{when } x < 0. \end{cases}$$

   (b) Note that

$$\lim_{h\to 0} \frac{|h|^3 - 0}{h - 0} = \lim_{h\to 0} \frac{|h|h^2}{h} = \lim_{h\to 0} |h|h = 0.$$

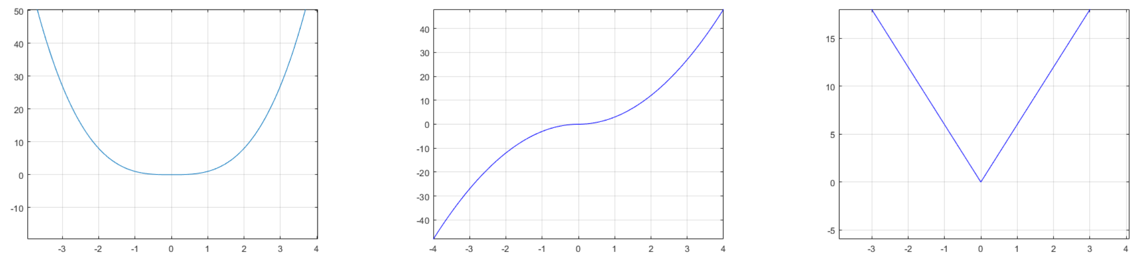
   Hence f is differentiable at $x = 0$ with $f'(x) = 0$.

(c) Note that, by (a) and (b),

$$\lim_{h\to 0^+} \frac{f'(h)-f'(0)}{h-0} = \lim_{h\to 0^+} \frac{3h^2}{h} = \lim_{h\to 0^+} 3h = 0.$$

$$\lim_{h\to 0^-} \frac{f'(h)-f'(0)}{h-0} = \lim_{h\to 0^-} \frac{-3h^2}{h} = \lim_{h\to 0^-} -3h = 0.$$

Hence $f'(x)$ is differentiable at $x = 0$ with $f''(x) = 0$.



(a) graph of f (b) graph of f' (c) graph of f''

Figure 2: Graph of Q3

4. Let

$$f(x) = \begin{cases} (x-2)^2 \sin\left(\dfrac{1}{x-2}\right), & \text{when } x \neq 2; \\ 0, & \text{when } x = 2. \end{cases}$$

(a) Is f continuous on $\mathbb{R}$?

(b) Is f differentiable on $\mathbb{R}$?

(c) Is f' continuous on $\mathbb{R}$?

Solution

(a) $\lim\limits_{x\to 2} f(x)$

$= \lim\limits_{x\to 2}(x-1)^2 \sin\left(\dfrac{1}{x-2}\right)$

$= 0$ (by squeeze theorem)

$= f(2)$

So f is continuous.

(b) $\lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$

$= \lim\limits_{x\to 2}(x-2) \sin\left(\dfrac{1}{x-2}\right)$

$= 0$ by squeeze theorem.

So $f'(2) = 0$.

When $x \neq 2$,

$f'(x)$

$= \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{1}{h}\left((x+h-2)^2 \sin\left(\dfrac{1}{x+h-2}\right) - (x-2)^2 \sin\left(\dfrac{1}{x-2}\right)\right)$

$= \lim\limits_{h\to 0} \dfrac{1}{h}\left((x-2)^2\left(\sin\left(\dfrac{1}{x+h-2}\right) - \sin\left(\dfrac{1}{x-1}\right)\right) + (2h(x-2)+h^2) \sin\left(\dfrac{1}{x+h-2}\right)\right)$

$= \left[\lim_{h\to 0} \frac{1}{h}(x-2)^2 \left(2\cos\left(\frac{x-2+h/2}{(x+h-2)(x-2)}\right)\sin\left(\frac{-h/2}{(x+h-2)(x-2)}\right)\right)\right] +$
$2(x-2)\sin\left(\frac{1}{x-2}\right)$
$= -\cos\left(\frac{1}{x-2}\right) + 2(x-2)\sin\left(\frac{1}{x-2}\right)$

So f is differentiable.

(c) $\lim_{x\to 2} f'(x)$ does not exist. So f' is not continuous.

5. Find natural domains of the following functions and differentiate them on their natural domains. You are not required to do so from first principles.

(a) $f(x) = \dfrac{1+\sin\ x}{1+2\cos\ x}$.

(b) $f(x) = (1+\tan^2 x)\cos^2 x$.

(c) $f(x) = \ln(\ln(\ln\ x))$

(d) $f(x) = \ln|\sin\ x|$

Solution

(a) Since $1+2\cos x \neq 0$, then $x \neq \frac{2}{3}\pi + 2k\pi$ and $x \neq \frac{4}{3}\pi + 2k\pi$, we know that the domain is $\left(\frac{2}{3}\pi + 2k\pi, \frac{4}{3}\pi + 2k\pi\right) \cup \left(\frac{4}{3}\pi + 2k\pi, \frac{8}{3}\pi + 2k\pi\right)$, $k \in \mathbb{Z}$.

$$\begin{aligned} f'(x) &= \frac{\cos x(1+2\cos x) + 2\sin x(1+\sin x)}{(1+2\cos x)^2} \\ &= \frac{2+2\sin x+\cos x}{(1+2\cos x)^2} \end{aligned}$$

(b) $\tan x$ is well-defined on $\mathbb{R}\setminus\{\frac{(2n-1)\pi}{2} : n \in \mathbb{Z}\}$. Therefore, this is also the natural domain of f.

Note that $f(x) = (1+\tan^2 x)\cos^2 x = \cos^2 x + \sin^2 x = 1$. Hence, $f'(x) = 0$.

(c)

$$\ln x > 0 \tag{1}$$
$$x > 1 \tag{2}$$
$$\ln(\ln x) > 0 \tag{3}$$
$$\ln x > 1 \tag{4}$$
$$x > e \tag{5}$$

By considering the intersection of the intervals above, the natural domain is given by (e, ∞).

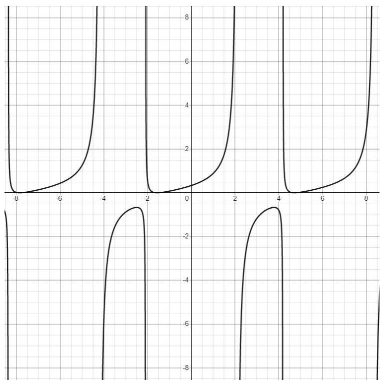
$$\begin{aligned} f'(x) &= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\ &= \frac{1}{x\ln x\ln(\ln x)} \end{aligned}$$

(d)

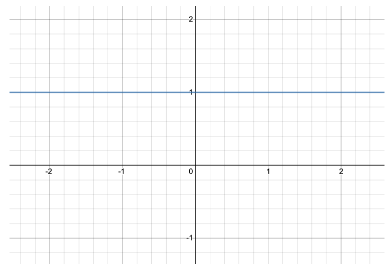
$$\begin{aligned}|\sin x| &> 0\\ \sin x &\neq 0\\ x &\neq n\pi, n \in \mathbb{Z}\end{aligned}$$

Therefore, the natural domain of f is $\mathbb{R}\backslash\{n\pi : n \in \mathbb{Z}\}$. Note that $f(x) = \ln(\pm \sin x)$. Therefore,

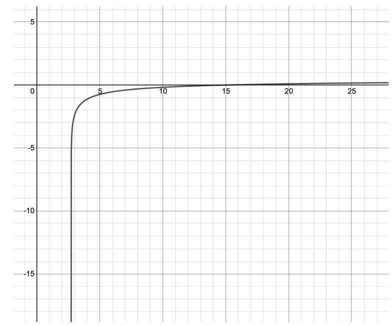
$$\begin{aligned}f'(x) &= \frac{1}{\pm \sin x} \cdot \pm \cos x\\ &= \frac{\cos x}{\sin x}\\ &= \cot x\end{aligned}$$



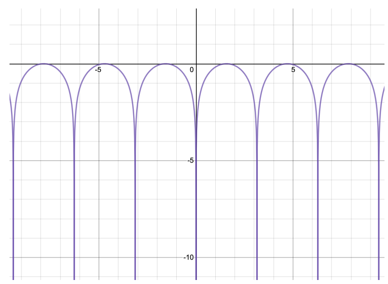
(a) 5a



(b) 5b



(c) 5c



(d) 5d

Figure 3: Graph of Q5

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Suppose f is differentiable at $x = 0$, with $f'(0) = a$. Show that $f(x) = ax$.

Solution
Let $x = y = 0$, we have

$$f(0) = 2f(0).$$

Hence $f(0) = 0$.
Since f is differentiable at $x = 0$, we have

$$a = f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}.$$

For each fixed $x \in \mathbb{R}$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) + f(h) - f(x)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = a.$$

This indicates that f is differentiable everywhere with $f'(x) = a$. Then $f(x) = ax+c$ for some $c \in \mathbb{R}$.
However, we must have $c = 0$ since $f(0) = c = 0$.

7. Find $\dfrac{dy}{dx}$ if

   (a) $x^2 + y^2 = e^{xy}$

(b) $x^3y + \sin(xy^2) = 4$

(c) $y = \tan^{-1}\sqrt{x}$

(d) $y = 3^{\sin x}$

(e) $y = x^{\ln x}$

(f) $y = x^{x^x}$

(g) $y = \sin(\cos(\sin x))$

Solution

(a) $x^2 + y^2 = e^{xy}$

$$2x + 2y\frac{dy}{dx} = \left(y + x\frac{dy}{dx}\right)e^{xy}$$

$$\frac{dy}{dx} = \frac{ye^{xy} - 2x}{2y - xe^{xy}}$$

(b) $x^3y + \sin xy^2 = 1$

$$3x^2y + x^3\frac{dy}{dx} + \left(y^2 + 2xy\frac{dy}{dx}\right)\cos xy^2 = 0$$

$$\frac{dy}{dx} = \frac{-3x^2y - y^2\cos xy^2}{x^3 + 2xy\cos xy^2}$$

(c) $y = \tan^{-1}\sqrt{x}$

$$\tan y = \sqrt{x}$$

$$\sec^2 y\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{\cos^2 y}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$$

(d) $y = 3^{\sin x}$

$$\frac{dy}{dx} = 3^{\sin x}(\ln 3)\cos x$$

(e) $y = x^{\ln x}$

$$\ln y = (\ln x)^2$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\ln x}{x}$$

$$\frac{dy}{dx} = \frac{2y\ln x}{x} = \frac{2x^{\ln x}\ln x}{x}$$

(f) $y = x^{x^x}$

$$\ln y = x^x \ln x$$

$$\ln\ln y = x\ln x + \ln\ln x$$

$$\frac{1}{y\ln y}\frac{dy}{dx} = \ln x + 1 + \frac{1}{x\ln x}$$

$$\frac{dy}{dx} = (y\ln y)\left(\ln x + 1 + \frac{1}{x\ln x}\right) = (x^{x^x}\cdot x^x\ln x)\left(\ln x + 1 + \frac{1}{x\ln x}\right)$$

(g) $y = \sin(\cos(\sin x))$
Let $g = \cos(\sin x)$,
we have $\frac{dg}{dx} = -\sin(\sin x) \cdot (\sin x)' = -\sin(\sin x) \cdot \cos x$.
Thus, $y = \sin(g)$,
$\frac{dy}{dx} = \cos(g) \cdot \frac{dg}{dx} = -\cos(\cos(\sin x)) \cdot \sin(\sin x) \cdot \cos x$

8. Find $\dfrac{d^2y}{dx^2}$ if

   (a) $y = \ln \tan x$

   (b) $y = \sin^{-1} \sqrt{1 - x^2}$

   (c) $y^2 = x^3 - x$

   (d) $\cos^2 y + \sin x = 1$

Solution

(a)

$$\frac{dy}{dx} = \frac{1}{\tan x} \cdot \sec^2 x = \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x} = 2\csc(2x)$$

$$\frac{d^2y}{dx^2} = -4\csc(2x)\cot(2x)$$

(b)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (\sqrt{1 - x^2})^2}} \cdot \frac{-2x}{2\sqrt{1 - x^2}} = -\frac{x}{\sqrt{x^2 - x^4}}$$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{x^2 - x^4} - x \cdot \frac{2x - 4x^3}{2\sqrt{x^2 - x^4}}}{x^2 - x^4} = -\frac{x^2 - x^4 - x(x - 2x^3)}{(x^2 - x^4)^{\frac{3}{2}}} = -\frac{x^4}{(x^2 - x^4)^{\frac{3}{2}}}$$

(c)

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2} = -\frac{y - x(-\frac{x}{y})}{y^2} = -\frac{x^2 + y^2}{y^3}$$

(d) Since $\cos^2 y + \sin x = 1 \Longrightarrow 2\cos y \cdot (-\sin y) \cdot \frac{dy}{dx} + \cos x = 0$, then

$$\frac{dy}{dx} = \frac{\cos x}{2\sin y \cos y} = \frac{\cos x}{\sin 2y}.$$

$$\Longrightarrow \sin 2y\frac{dy}{dx} = \cos x.$$

Hence

$$
\begin{aligned}
2\cos 2y\left(\frac{dy}{dx}\right)^2+\sin 2y\cdot\frac{d^2y}{dx^2}&=-\sin x\\
\sin 2y\frac{d^2y}{dx^2}&=-\sin x-2\cos 2y\cdot\frac{\cos^2 x}{\sin^2 2y}\\
\frac{d^2y}{dx^2}&=-\frac{\sin x}{\sin 2y}-\frac{2\cot 2y\cos^2 x}{\sin^2 2y}
\end{aligned}
$$

Therefore

$$
\frac{d^2y}{dx^2}=-\frac{\sin x\sin 2y+2\cot 2y\cos^2 x}{\sin^2 2y}.
$$

9. Find the n-th derivative of the following functions for all positive integers n.

   (a) $f(x)=(e^x+e^{-x})^2$, $x\in\mathbb{R}$

   (b) $f(x)=\dfrac{1}{1-x^2}$, $x\in(-1,1)$

   (c) $f(x)=\sin x\cos x$, $x\in\mathbb{R}$

   (d) $f(x)=\cos^2 x$, $x\in\mathbb{R}$

   (e) $f(x)=\dfrac{x^2}{e^x}$, $x\in\mathbb{R}$

Solution

(a) Simplify $f(x)$ first,

$$
f(x)=(e^x+e^{-x})^2=e^{2x}+2+e^{-2x}.
$$

Hence,

$$
f^{(n)}(x)=2^ne^{2x}+(-2)^ne^{-2x}.
$$

(b) Process the partial fraction for $f(x)$. Suppose

$$
f(x)=\frac{A}{1+x}+\frac{B}{1-x},
$$

where A,B is a constant, then we have

$$
\frac{1}{1-x^2}=\frac{(B-A)x+(B+A)}{1-x^2},
$$

by comparing the coefficients, we have

$$
\begin{cases}
B+A &=1,\\
B-A &=0.
\end{cases}
$$

Hence, $A = B = \dfrac{1}{2}$, and

$$f(x) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right).$$

Therefore,

$$f^{(n)}(x) = \frac{1}{2}\left[(-1)^n \frac{n!}{(1+x)^{n+1}} + \frac{n!}{(1-x)^{n+1}}\right].$$

(c) By double angle formula,

$$f(x) = \sin x \cos x = \frac{1}{2}\sin 2x.$$

Hence,

$$f^{(n)}(x) = \begin{cases} 2^{n-1}\sin 2x & \text{if } n = 4k \text{ for some } k \in \mathbb{N}, \\ 2^{n-1}\cos 2x & \text{if } n = 4k+1 \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\sin 2x & \text{if } n = 4k+2 \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\cos 2x & \text{if } n = 4k+3 \text{ for some } k \in \mathbb{N}. \end{cases}$$

(d) By double angle formula,

$$f(x) = \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Hence,

$$f^{(n)}(x) = \begin{cases} 2^{n-1}\cos 2x & \text{if } n = 4k \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\sin 2x & \text{if } n = 4k+1 \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\cos 2x & \text{if } n = 4k+2 \text{ for some } k \in \mathbb{N}, \\ 2^{n-1}\sin 2x & \text{if } n = 4k+3 \text{ for some } k \in \mathbb{N}. \end{cases}$$

(e) Note that

$$f(x) = \frac{x^2}{e^x} = x^2 e^{-x} = g(x)h(x)$$

where $g(x) = x^2$, $h(x) = e^{-x}$. Using Leibniz Rule (proved by mathematical induction and product rule),

$$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(x)\, h^{(n-k)}(x).$$

Note that $g'(x) = 2x$, $g''(x) = 2$ and $g^{(k)}(x) = 0$ for all $k \geq 3$. Hence,

$$\begin{aligned} f^{(n)}(x) &= \binom{n}{0} g(x)\, h^{(n)}(x) + \binom{n}{1} g'(x)\, h^{(n-1)}(x) + \binom{n}{2} g''(x)\, h^{(n-2)}(x) \\ &= (-1)^n x^2 e^{-x} + (-1)^{n+1} 2nxe^{-x} + (-1)^n n(n-1)e^{-x}. \end{aligned}$$

10. Find all points (x_0, y_0) on the graph of

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 8$$

where lines tangent to the graph at (x_0, y_0) have slope -1.

Solution

We differentiate both sides of the equation and get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0.$$

Thus,

$$y' = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

Since $y' = -1$ at (x_0, y_0), we have

$$y_0^{\frac{1}{3}} = x_0^{\frac{1}{3}},$$

and thus $x_0 = y_0$. Plugging this back to the equation, we have

$$2x_0^{\frac{2}{3}} = 8,$$

and so $x_0 = \pm 8$. Therefore, $(x_0, y_0) = (8, 8)$ or $(-8, -8)$.

11. The chain rule says

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

or equivalently,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where $y = f(u)$ and $u = g(x)$.

(a) Give examples to show

$$(f \circ g)''(x) \neq f''(g(x)) \cdot g''(x),$$

or equivalently,

$$\frac{d^2y}{dx^2} \neq \frac{d^2y}{du^2} \cdot \frac{d^2u}{dx^2},$$

where $\frac{d^2y}{dx^2}$ denotes the second derivative of $y = f(x)$.

(b) Prove

$$(f \circ g)''(x) = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x).$$

Solution

(a) Let $y = u^2$ and $u = x$.
Then $y = x^2$.
$\frac{dy}{dx} = 2x$
$\frac{d^2y}{dx^2} = 2$
$\frac{d^2u}{dx^2} = 0$
$\frac{d^2y}{du^2} \cdot \frac{d^2u}{dx^2} = 0$

(b) Prove

$$(f \circ g)''(x) = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x).$$

Solution

$y = f(u)$ and $u = g(x)$.
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\frac{d}{dx}\frac{dy}{dx}$
$= \frac{d}{dx}\left(\frac{dy}{du} \cdot \frac{du}{dx}\right)$
$= \frac{d}{dx}\left(\frac{dy}{du}\right) \cdot \frac{du}{dx} + \frac{dy}{du} \cdot \frac{d^2u}{dx^2}$
$= \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du} \cdot \frac{d^2u}{dx^2}$

12. (a) Suppose $a, b > 0$ are constants, and

$$y = \frac{1}{ab} \arctan\left(\frac{b}{a} \tan x\right)$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Express $\frac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.

(b) Suppose $a, b > 0$ are constants, and

$$y = \ln\left|\frac{a + b\tan x}{a - b\tan x}\right|$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left\{\pm \arctan\left(\frac{a}{b}\right)\right\}$. Express $\frac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.

Solution

(a)

$$\frac{dy}{dx} = \frac{1}{ab}\frac{1}{1 + (\frac{b}{a}\tan x)^2} \cdot \frac{b}{a}\sec^2 x = \frac{1}{a^2\cos^2 x + b^2 \sin^2 x}$$

(b)Note that

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad \text{for } x \neq 0.$$

Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{a\cos x - b\sin x}{a\cos x + b\sin x}\cdot\frac{d}{dx}\left(\frac{a\cos x + b\sin x}{a\cos x - b\sin x}\right)\\ &= \frac{a\cos x - b\sin x}{a\cos x + b\sin x}\cdot\frac{(a\cos x - b\sin x)(-a\sin x + b\cos x) - (a\cos x + b\sin x)(-a\sin x - b\cos x)}{(a\cos x - b\sin x)^2}\\ &= \frac{2ab}{a^2\cos^2 x - b^2\sin^2 x}\end{aligned}$$

13. Let a, b be real numbers and $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{3 + a\sqrt{1 - \frac{2}{3}x}}{x} & \text{if } x < 0 \\ 1 + b\tan\left(\dfrac{x}{10}\right) & \text{if } x \geq 0. \end{cases}$$

Assume that $f(x)$ is continuous at $x = 0$.

(a) Determine the value of a and justify your computation.

(b) If we further assume that f is differentiable at 0, determine the value of b and justify your answer.

Solution

(a) Since function f is piecewise-defined and continuous at $x = 0$, we need to consider the two-side limit of f at 0, and check that the following equality holds

$$\lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x)$$

On one hand by definition, for $x \geq 0$, $f(x) = 1 + b\tan(x/10)$, so we have $f(0) = 1 + 0 = 1$, and the righthand side limit at 0 matches the value of f at 0 no matter what value b takes:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}[1 + b\tan(x/10)] = 1 = f(0).$$

On the other hand, the lefthand side limit

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{3 + a\sqrt{1 - 2x/3}}{x}$$

must exist and equal to 1. Note that the denominator of the fraction above is just x, which tends to 0 as $x \to 0^-$ while the numerator is $3 + a\sqrt{1 - 2x/3}$ which tends to $3+a$ as $x \to 0^-$, so in order that the limit exists, the numerator $3 + a\sqrt{1 - 2x/3}$ must tends to 0 as well, otherwise the limit would look like "$\frac{\text{some nonzero number}}{0}$" which means the limit could not exist. We have $3 + a = 0$, $a = -3$. And one can check that the equality holds

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{3 - 3\sqrt{1 - 2x/3}}{x} = 1 = f(0).$$

(b) Since f is differentiable at 0, both limits

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

should exist and equal to each other. For the lefthand side limit

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{1}{h}\left(\frac{3-3\sqrt{1-2h/3}}{h} - 1\right) \\
&= \lim_{h\to 0^-} \frac{3-h-3\sqrt{1-2h/3}}{h^2} \\
&= \lim_{h\to 0^-} \frac{(3-h)^2 - 9(\sqrt{1-2h/3})^2}{h^2[(3-h)+3\sqrt{1-2h/3}]} \\
&= \lim_{h\to 0^-} \frac{(3-h)^2 - 9(1-2h/3)}{h^2[(3-h)+3\sqrt{1-2h/3}]} \\
&= \lim_{h\to 0^-} \frac{1}{(3-h)+3\sqrt{1-2h/3}} = \frac{1}{6}.
\end{aligned}$$

while for the righthand side limit

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+} \frac{[1+b\tan(h/10)]-1}{h} \\
&= \lim_{h\to 0^+} \frac{b}{10}\frac{\sin(h/10)}{h/10}\frac{1}{\cos(h/10)} = \frac{b}{10}.
\end{aligned}$$

Therefore $\frac{b}{10} = \frac{1}{6}$ and hence $b = \frac{5}{3}$.