THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2022-2023 Term 1
Suggested Solutions of Homework Assignment 1

1. Determine the limit of each of the following sequences, or show that the sequence diverges. You may make use of the limit laws and theorems covered in class.

   (a) $a_n = \dfrac{3n-7}{n+2} - \left(\dfrac{4}{5}\right)^n \qquad \text{for } n \geq 1.$

   (b) $a_n = \sqrt{n}(\sqrt{n+4} - \sqrt{n}) \qquad \text{for } n \geq 1.$

   (c) $a_n = \dfrac{7^n}{n!} \qquad \text{for } n \geq 1.$

   (d) $a_n = \dfrac{\sin(n^2)}{n} \qquad \text{for } n \geq 1.$

   (e) $a_n = \dfrac{n}{n + n^{1/n}} \qquad \text{for } n \geq 1.$

   (f) $a_n = \left(3 + \dfrac{2}{n^2}\right)^{1/3} \qquad \text{for } n \geq 1.$

   Solutions:

   (a)

$$\begin{aligned} a_n &= \frac{3n-7}{n+2} - \left(\frac{4}{5}\right)^n \quad \text{for } n \geq 1 \\ \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left[\frac{3n-7}{n+2} - \left(\frac{4}{5}\right)^n\right] \\ &= \lim_{n\to\infty} \left[\frac{3-\frac{7}{n}}{1+\frac{2}{n}} - \left(\frac{4}{5}\right)^n\right] \\ &= \frac{3-0}{1+0} - 0 \\ &= 3 \end{aligned}$$

(b)

$$
\begin{aligned}
a_n &= \sqrt{n}\left(\sqrt{n+4}-\sqrt{n}\right) \text{ for } n \geq 1 \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+4}-\sqrt{n}\right) \cdot \frac{\sqrt{n+4}+\sqrt{n}}{\sqrt{n+4}+\sqrt{n}} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}\cdot(n+4-n)}{\sqrt{n+4}+\sqrt{n}} \\
&= \lim_{n\to\infty} \frac{1\cdot 4}{\sqrt{1+\frac{4}{n}}+1} \\
&= \frac{4}{\sqrt{1+0}+1} \\
&= 2
\end{aligned}
$$

(c)

$$
a_n = \frac{7^n}{n!} \text{ for } n \geq 1
$$

Note that for $n > 7$,

$$
\begin{aligned}
a_n &= \frac{7^7}{7!}\cdot\frac{7}{8}\cdot\frac{7}{9}\cdot \ldots \cdot \frac{7}{n} \\
&< \frac{7^7}{7!}\cdot 1 \cdot\ 1 \cdot\ \ldots \cdot\ \frac{7}{n} \\
&= \frac{7^8}{7!}\cdot\frac{1}{n}
\end{aligned}
$$

Then for $n > 7$, We have

$$
0 < a_n < \frac{7^8}{7!}\cdot\frac{1}{n}
$$

Since $\lim\limits_{n\to\infty} \frac{7^8}{7!}\cdot\frac{1}{n} = 0$, by sandwich theorem, $\lim\limits_{n\to\infty} a_n = 0$.

(d)

$$
a_n = \frac{\sin\, n^2}{n} \text{ for } n \geq 1
$$

We have $-1 \leq \sin n^2 \leq 1$

Then $\frac{-1}{n} \leq \frac{\sin n^2}{n} \leq \frac{1}{n}$

Since $\lim\limits_{n\to\infty} \frac{-1}{n} = 0$ and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$,

by sandwich theorem, $\lim\limits_{n\to\infty} a_n = 0$.

(e)

$$a_n = \frac{n}{n + n^{1/n}} \text{ for } n \geq 1$$

We first prove that $0 < n^{1/n} < 2$.
Clearly, $n^{1/n} > 0$ since n is positive.
We can use mathematical induction to prove that $n < 2^n$, hence $n^{1/n} < 2$.
For $n = 1, 2^1 = 2 > 1$
For $n = k + 1, k + 1 \leq 2k < 2 \cdot 2^k = 2^{k+1}$
Then $0 < n^{1/n} < 2$.

$$\frac{n}{n+2} < \frac{n}{n + n^{1/n}} < \frac{n}{n+0} = 1$$

Since $\lim\limits_{n\to\infty} \frac{n}{n+2} = 1$,
by sandwich theorem, $\lim\limits_{n\to\infty} a_n = 1$.

(f)

$$a_n = \left(3 + \frac{2}{n^2}\right)^{1/3} \text{ for } n \geq 1$$

$$\begin{aligned}\lim_{n\to\infty} a_n &= (3+0)^{1/3} \\ &= 3^{1/3}\end{aligned}$$

2. Consider the following bounded and increasing sequence:

$$\begin{cases} a_1 = \sqrt{3} \\ a_2 = \sqrt{3+\sqrt{3}} \\ a_3 = \sqrt{3+\sqrt{3+\sqrt{3}}} \\ \vdots \\ a_{n+1} = \sqrt{3+a_n} \\ \vdots \end{cases}$$

Answer the following questions:

(a) Show that the sequence converges and find its limit.
(b) Answer the same question when 3 is replaced by an arbitrary integer $k \geq 2$.

Solutions:

(a) (i) Let $P(n)$ be the statement that $a_{n+1} \geq a_n$.

- When $n = 1$,

$$a_2 = \sqrt{3+\sqrt{3}} > \sqrt{3} = a_1$$

Hence, $P(1)$ is true.

- Suppose $P(m)$ is true, i.e.

$$a_{m+1} \geq a_m$$

- When $n = m + 1$,

$$a_{m+2} = \sqrt{3 + a_{m+1}} \geq \sqrt{3 + a_m} = a_{m+1}$$

  Hence, $P(m + 1)$ is true.

Therefore, $P(n)$ is true for any $n \geq 1$, i.e. $\{a_n\}$ is increasing.

(ii) Let $Q(n)$ be the statement that $a_{n+1} \leq \frac{1+\sqrt{13}}{2}$.

- When $n = 1$,

$$a_1 = \sqrt{3} < \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2} < \frac{1 + \sqrt{13}}{2}$$

  Hence, $Q(1)$ is true.

- Suppose $Q(m)$ is true, i.e.

$$a_m \leq \frac{1 + \sqrt{13}}{2}$$

- When $n = m + 1$,

$$a_{m+1} = \sqrt{3 + a_m} \leq \sqrt{3 + \frac{1 + \sqrt{13}}{2}} = \frac{\sqrt{1 + 2\sqrt{13} + 13}}{2} = \frac{1 + \sqrt{13}}{2}$$

  Hence, $Q(m + 1)$ is true.

Therefore, $Q(n)$ is true for any $n \geq 1$, i.e. $a_n \leq \frac{1+\sqrt{13}}{2}$.

By Monotone Convergence Theorem, $\{a_n\}$ is convergent.
Suppose $\lim_{n\to\infty} a_n = L$.

$$a_{n+1} = \sqrt{3 + a_n}$$
$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3 + a_n}$$
$$L = \sqrt{3 + L}$$
$$L^2 - L - 3 = 0$$
$$L = \frac{1 + \sqrt{13}}{2} \quad \text{or} \quad L = \frac{1 - \sqrt{13}}{2}$$

$L = \frac{1-\sqrt{13}}{2}$ is rejected since $a_n > 0$ for all n. Hence, $\lim_{n\to\infty} a_n = \frac{1+\sqrt{13}}{2}$.

(b) For any integer $k \geq 2$,

(i) Let $P(n)$ be the statement that $a_{n+1} \geq a_n$.

- When $n = 1$,

$$a_2 = \sqrt{k + \sqrt{k}} > \sqrt{k} = a_1$$

  Hence, $P(1)$ is true.

- Suppose $P(m)$ is true, i.e.

$$a_{m+1} \geq a_m$$

- When $n = m+1$,

$$a_{m+2} = \sqrt{k + a_{m+1}} \geq \sqrt{k + a_m} = a_{m+1}$$

Hence, $P(m+1)$ is true.

Therefore, $P(n)$ is true for any $n \geq 1$, i.e. $\{a_n\}$ is increasing.

(ii) Let $Q(n)$ be the statement that $a_{n+1} \leq \frac{1+\sqrt{1+4k}}{2}$.

- When $n = 1$,

$$a_1 = \sqrt{k} < \sqrt{\frac{1+4k}{4}} = \frac{\sqrt{1+4k}}{2} < \frac{1+\sqrt{1+4k}}{2}$$

Hence, $Q(1)$ is true.

- Suppose $Q(m)$ is true, i.e.

$$a_m \leq \frac{1+\sqrt{1+4k}}{2}$$

- When $n = m+1$,

$$a_{m+1} = \sqrt{k + a_m} \leq \sqrt{k + \frac{1+\sqrt{1+4k}}{2}} = \frac{\sqrt{1+2\sqrt{1+4k}+1+4k}}{2} = \frac{1+\sqrt{1+4k}}{2}$$

Hence, $Q(m+1)$ is true.

Therefore, $Q(n)$ is true for any $n \geq 1$, i.e. $a_n \leq \frac{1+\sqrt{1+4k}}{2}$.

By Monotone Convergence Theorem, $\{a_n\}$ is convergent.
Suppose $\lim_{n\to\infty} a_n = L$.

$$\begin{aligned} a_{n+1} &= \sqrt{k + a_n} \\ \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{k + a_n} \\ L &= \sqrt{k + L} \\ L^2 - L - k &= 0 \\ L = \frac{1+\sqrt{1+4k}}{2} \quad &\text{or} \quad L = \frac{1-\sqrt{1+4k}}{2} \end{aligned}$$

$L = \frac{1-\sqrt{1+4k}}{2}$ is rejected since $a_n > 0$ for all n. Hence, $\lim_{n\to\infty} a_n = \frac{1+\sqrt{1+4k}}{2}$.

3. For this problem, you may make use of the following mathematical result:

Fact. Let a, r be real numbers, with $r \neq 1$. Let $\{S_n\}$ be the geometric series defined as follows:

$$S_n = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n, \quad n = 0, 1, 2, \ldots.$$

Then, $S_n = a\left(\frac{1-r^{n+1}}{1-r}\right)$.

(a) Verify that $\{S_n\}$ converges to $\dfrac{a}{1-r}$, whenever $|r| < 1$.

(b) Use the result of Part (a) to find the limit of the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{3}{4} + \frac{3}{4^2} + \cdots + \frac{3}{4^n}.$$

(c) Use the result of Part (a) to verify that the repeating decimal $1.777\cdots$, often written as $1.\dot{7}$, is equal to $\dfrac{16}{9}$.

Solutions:

(a) When $|r| < 1$, we have $1 - r \neq 0$ and $\lim\limits_{n\to\infty} r^{n+1} = 0$.
Then $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} a\left(\frac{1-r^{n+1}}{1-r}\right) = a\left(\frac{1-\lim\limits_{n\to\infty} r^{n+1}}{1-r}\right) = a\left(\frac{1-0}{1-r}\right) = \frac{a}{1-r}$.

(b) Let $a = 3$ and $r = \frac{1}{4}$. Then $a_n = S_n - 2$.
Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} S_n - 2 = \frac{a}{1-r} - 2 = \frac{3}{1-\frac{1}{4}} - 2 = 2$.

(c) Let $a = 7$ and $r = \frac{1}{10}$. Then $a_n = S_n - 6$.
Then $1.\dot{7} = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} S_n - 6 = \frac{a}{1-r} - 6 = \frac{7}{1-\frac{1}{10}} - 6 = \frac{16}{9}$.

4. A sequence $\{a_n\}$ is defined recursively by the following equations:

$$\begin{cases} a_1 = 1, \\ a_{n+1} = \sqrt{7 + 2a_n} \quad \text{for } n \geq 1. \end{cases}$$

Answer the following questions:

(a) Show that $\{a_n\}$ is bounded and monotonic and hence convergent.

(b) Find the limit of $\{a_n\}$.

Solutions:

(a) (i) Let $P(n)$ be the statement that $a_{n+1} \geq a_n$.

- When $n = 1$,

$$a_2 = \sqrt{7+2} = 3 > 1 = a_1$$

Hence, $P(1)$ is true.

- Suppose $P(m)$ is true, i.e.

$$a_{m+1} \geq a_m$$

- When $n = m + 1$,

$$a_{m+2} = \sqrt{7 + 2a_{m+1}} \geq \sqrt{7 + 2a_m} = a_{m+1}$$

Hence, $P(m+1)$ is true.

Therefore, $P(n)$ is true for any $n \geq 1$, i.e. $\{a_n\}$ is increasing.

(ii) Let $Q(n)$ be the statement that $a_{n+1} \leq 1 + 2\sqrt{2}$.

- When $n = 1$,

$$a_1 = 1 < 1 + 2\sqrt{2}$$

  Hence, $Q(1)$ is true.

- Suppose $Q(m)$ is true, i.e.

$$a_m \leq 1 + 2\sqrt{2}$$

- When $n = m + 1$,

$$a_{m+1} = \sqrt{7 + 2a_m} \leq \sqrt{7 + 2 + 4\sqrt{2}} = \sqrt{1 + 2 \times 2\sqrt{2} + 8} = 1 + 2\sqrt{2}$$

  Hence, $Q(m + 1)$ is true.

Therefore, $Q(n)$ is true for any $n \geq 1$, i.e. $a_n \leq 1 + 2\sqrt{2}$.

By Monotone Convergence Theorem, $\{a_n\}$ is convergent.

(b) Suppose $\lim_{n\to\infty} a_n = L$.

$$\begin{aligned} a_{n+1} &= \sqrt{7 + 2a_n} \\ \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{7 + 2a_n} \\ L &= \sqrt{7 + 2L} \\ L^2 - 2L - 7 &= 0 \end{aligned}$$

$$L = 1 + 2\sqrt{2} \quad \text{or} \quad L = 1 - 2\sqrt{2}$$

$L = 1 - 2\sqrt{2}$ is rejected since $a_n > 0$ for all n. Hence, $\lim_{n\to\infty} a_n = 1 + 2\sqrt{2}$.

5. Let $k > 0$ and a_1 be a positive number. Define a sequence $\{a_n\}$ by the relation:

$$a_{n+1} = \sqrt{k + a_n} \quad \text{for } n \geq 1.$$

Let α be the positive root of the equation:

$$x^2 - x - k = 0.$$

(a) Suppose $0 < a_1 < \alpha$. Show that the sequence $\{a_n\}$ is monotonic increasing and converges to α.

(b) Suppose $a_1 > \alpha$. Show that the sequence $\{a_n\}$ is monotonic decreasing and converges to α.

Solution:

(a) Let $P(n)$ be the statement that $a_{n+1} \geq a_n$.

- First we note that $x^2 - x - k = 0$ has a positive root α and a negative root $-k/\alpha$, and that $x^2 - x - k < 0$ whenever $-k/\alpha < x < \alpha$.
  Since $0 < a_1 < \alpha$, we have $a_1^2 - a_1 - k < 0$, and so $a_1 < \sqrt{k + a_1} = a_2$.
  Hence, $P(1)$ is true.
- Suppose $P(m)$ is true, i.e. $a_{m+1} \geq a_m$.
- When $n = m + 1$,

$$a_{m+2} = \sqrt{k + a_{m+1}} \geq \sqrt{k + a_m} = a_{m+1}.$$

  Hence, $P(m + 1)$ is true.

By mathematical induction, $P(n)$ is true for all $n \geq 1$, i.e. $\{a_n\}$ is monotonic increasing.

Next, we show that $\{a_n\}$ is bounded above by α. Let $Q(n)$ be the statement that $a_n < \alpha$.

- Clearly, $a_1 < \alpha$. Hence, $Q(1)$ is true.
- Suppose $Q(m)$ is true, i.e. $a_m < \alpha$.
- When $n = m + 1$,

$$a_{m+1} = \sqrt{k + a_m} < \sqrt{k + \alpha} = \sqrt{\alpha^2} = \alpha.$$

  Hence, $Q(m + 1)$ is true.

By mathematical induction, $Q(n)$ is true for all $n \geq 1$. So $\{a_n\}$ is bounded above by α.

By Monotone Convergence Theorem, $\{a_n\}$ converges. Let $\ell = \lim_{n \to +\infty} a_n$. Then

$$\begin{aligned} \lim_{n \to +\infty} a_{n+1}^2 &= \lim_{n \to +\infty} (k + a_n) \\ \ell^2 - \ell - k = 0. \end{aligned}$$

Since $a_n \geq a_1 > 0$ for all $n \geq 1$, we have $\ell \geq a_1 > 0$.
So ℓ is the positive root of $x^2 - x - k = 0$. Therefore, $\lim_{n \to +\infty} a_n = \ell = \alpha$.

(b) Let $P(n)$ be the statement that $a_{n+1} < a_n$ and $a_n > \alpha$.

- Since $a_1 > \alpha$, we have $a_1^2 - a_1 - k > 0$, and so $a_1 > \sqrt{k + a_1} = a_2$.
  Hence, $P(1)$ is true.
- Suppose $P(m)$ is true, i.e. $a_{m+1} < a_m$ and $a_m > \alpha$..
- When $n = m + 1$,

$$a_{m+2} = \sqrt{k + a_{m+1}} < \sqrt{k + a_m} = a_{m+1},$$

  and

$$a_{m+1} = \sqrt{k + a_m} > \sqrt{k + \alpha} = \sqrt{\alpha^2} = \alpha.$$

  Hence, $P(m + 1)$ is true.

By mathematical induction, $P(n)$ is true for all $n \geq 1$. Thus, $\{a_n\}$ is monotonic decreasing and bounded below by α.
By Monotone Convergence Theorem, $\{a_n\}$ converges. Let $\ell = \lim_{n\to+\infty} a_n$. Then

$$\begin{aligned}\lim_{n\to+\infty} a_{n+1}^2 &= \lim_{n\to+\infty} (k + a_n)\\ \ell^2 - \ell - k &= 0.\end{aligned}$$

Since $a_n > \alpha > 0$ for all $n \geq 1$, we have $\ell \geq \alpha > 0$.
So ℓ is the positive root of $x^2 - x - k = 0$. Therefore, $\lim_{n\to+\infty} a_n = \ell = \alpha$.

6. Given a sequence $\{a_n\}$ such that $a_1 > a_2 > 0$, and

$$a_{n+2} = \frac{1}{2}(a_{n+1} + a_n), \quad \text{for} \quad n = 1, 2, \cdots .$$

Answer the following questions:

(a) Show that for $n \geq 1$,

$$a_{n+2} - a_n = \frac{(-1)^n}{2^n}(a_1 - a_2)$$

and hence show that the sequence $\{a_1, a_3, a_5, \cdots\}$ is strictly decreasing and that the sequence $\{a_2, a_4, a_6, \cdots\}$ is strictly increasing.

(b) For any positive integers m and n, show that

$$a_{2m} < a_{2n-1}.$$

(c) Show that the two sequences $\{a_1, a_3, a_5, \cdots\}$ and $\{a_2, a_4, a_6, \cdots\}$ converge to the same limit k, where

$$k = \frac{1}{3}(a_1 + 2a_2).$$

Solution:

(a) Because

$$a_{n+1} - a_n = \frac{1}{2}\left(a_n + a_{n-1}\right) - a_n = -\frac{1}{2}\left(a_n - a_{n-1}\right),$$

we have

$$\begin{aligned}a_{n+1} - a_n &= -\frac{1}{2}\left(a_n - a_{n-1}\right)\\ &= (-\frac{1}{2})^2\left(a_{n-1} - a_{n-2}\right)\\ &= (-\frac{1}{2})^3\left(a_{n-2} - a_{n-3}\right)\\ &\vdots\\ &= (-\frac{1}{2})^{n-1}\left(a_2 - a_1\right).\end{aligned}$$

Hence,

$$\begin{aligned} a_{n+2} - a_n &= \frac{1}{2}\left(a_{n+1} + a_n\right) - a_n \\ &= \frac{1}{2}\left(a_{n+1} - a_n\right) \\ &= \frac{1}{2}(-\frac{1}{2})^{n-1}\left(a_2 - a_1\right) \\ &= (-\frac{1}{2})^{n}\left(a_2 - a_1\right). \end{aligned}$$

Since $a_2 - a_1 > 0$, it follows that $a_{n+2} - a_n \begin{cases} > 0 & \text{when } n \text{ is even} \\ < 0 & \text{when } n \text{ is odd} \end{cases}$.

Accordingly, $\{a_{2n+1}\}$ is strictly decreasing and $\{a_{2n}\}$ is strictly increasing.

(b) For any $m, n \geq 1$, consider the following 3 cases:

(i) Let $m = n$. By (a), $2a_{2m} = a_{2m-1} + a_{2m-2} < a_{2m-1} + a_{2m}$. So $a_{2m} < a_{2m-1}$.

(ii) Let $m < n$. By (a) and (b)(i), $a_{2m} < a_{2n} < a_{2n-1}$.

(iii) Let $m > n$. By (a) and (b)(i), $a_{2n-1} > a_{2m-1} > a_{2m}$.

In all cases, $a_{2m} < a_{2n-1}$ for $m, n \geq 1$.

(c) By (a) and (b), $\{a_{2n+1}\}$ is decreasing and bounded below, e.g. by a_2, $\{a_{2n}\}$ is increasing and bounded above, e.g. by a_1. So, by Monotone Convergence Theorem, both sequences converge. Let $\lim\limits_{n\to\infty} a_{2n} = \ell_1$ and $\lim\limits_{n\to\infty} a_{2n+1} = \ell_2$. Then $\lim\limits_{n\to\infty} a_{n+2} = \lim\limits_{n\to\infty} \frac{1}{2}(a_{n+1} + a_n)$ implies that

$$\begin{cases} \ell_2 = \frac{1}{2}(\ell_1 + \ell_2) & \text{if } n \text{ is odd} \\ \ell_1 = \frac{1}{2}(\ell_2 + \ell_1) & \text{if } n \text{ is even}. \end{cases}$$

Thus, $\ell_1 = \ell_2$, i.e. $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} a_{2n+1}$.

Now, from the definition of the sequence,

$$\begin{aligned} \sum_{k=3}^{n} a_k &= \frac{1}{2}\sum_{k=3}^{n}(a_{k-2} + a_{k-1}) \\ &= \frac{1}{2}a_1 + \sum_{k=2}^{n-2} a_k + \frac{1}{2}a_{n-1} \\ \frac{1}{2}a_{n-1} + a_n &= \frac{1}{2}a_1 + a_2. \end{aligned}$$

Taking limit,

$$\begin{aligned} \frac{3}{2}\lim_{n\to\infty} a_n &= \frac{1}{2}a_1 + a_2 \\ \lim_{n\to\infty} a_n &= \frac{1}{3}(a_1 + 2a_2). \end{aligned}$$

7. For each of the given functions, f, find its natural domain, that is, the largest subset of $\mathbb{R}$ on which the expression defining f may be validly computed. Please express your answer in the form of a single interval, or a union of disjoint intervals. For example: $(-\infty, 2) \cup [5, 11)$.

(a) (Optional) $f(x) = \frac{1}{2}\sqrt{4 - x^2}$.

(b) $f(x) = \sqrt{\frac{x-2}{x+2}}$.

(c) (Optional) $f(x) = \ln(3x^2 - 4x + 5)$.

(d) $f(x) = \ln(\sqrt{x-4} + \sqrt{6-x})$.

(e) (Optional) $f(x) = \sin^2 x + \cos^4 x$.

(f) $f(x) = \frac{1}{1 + \cos x}$.

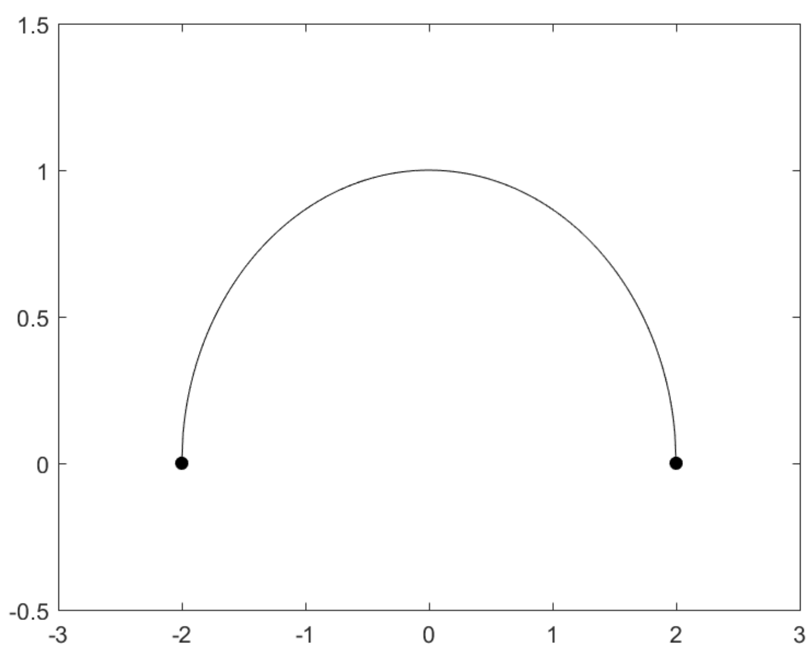
(g) $f(x) = 1 - |x|$.

Solutions:

(a)

$$f(x) = \frac{1}{2}\sqrt{4 - x^2}$$

It implies the condition $4 - x^2 \geq 0, -2 \leq x \leq 2$.
Hence the largest domain is $[-2, 2]$



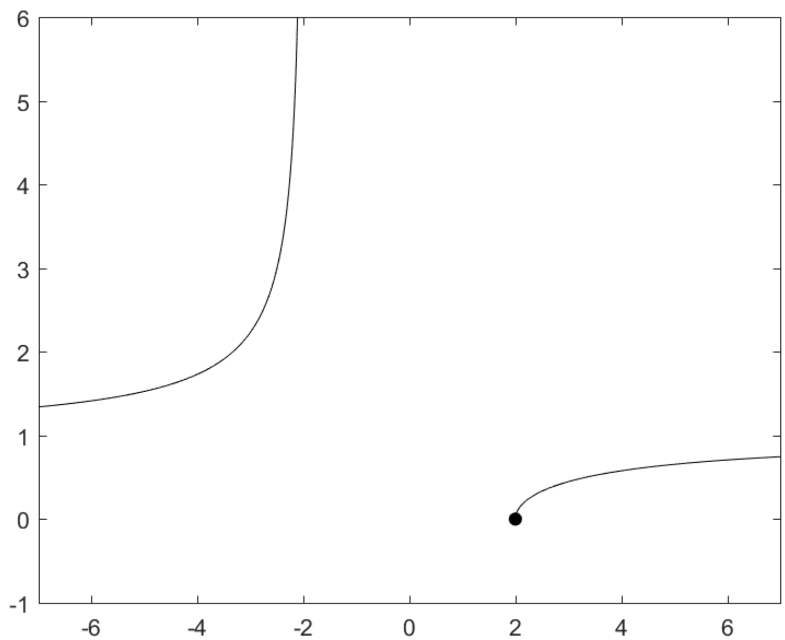
(b)

$$f(x) = \sqrt{\frac{x-2}{x+2}}$$

It implies two conditions $x \neq -2$ and $\dfrac{x-2}{x+2} \geq 0$.
For $\dfrac{x-2}{x+2} \geq 0$,

$$\begin{aligned} \frac{x-2}{x+2} &\geq 0 \\ \frac{x-2}{x+2} \cdot (x+2)^2 &\geq 0 \\ (x-2)(x+2) &\geq 0 \\ x \leq -2 &\text{ or } x \geq 2 \end{aligned}$$

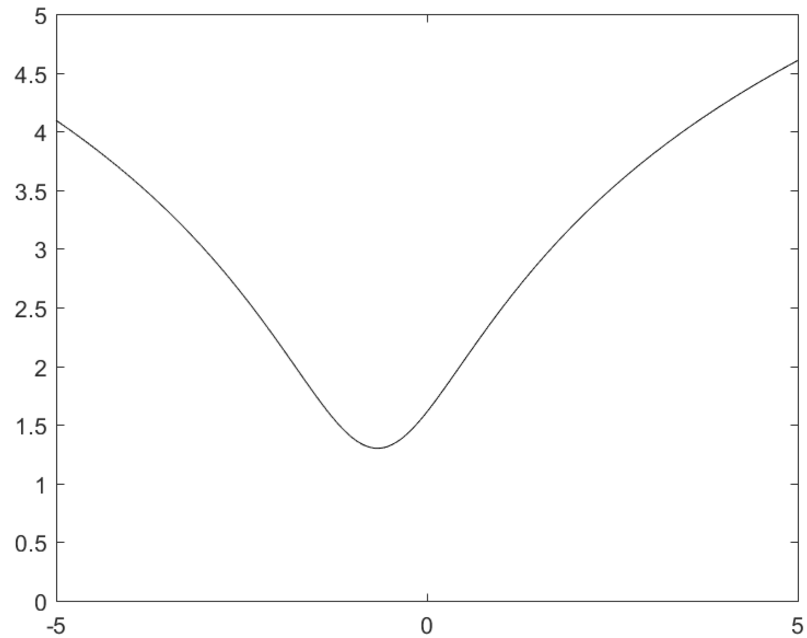
Hence the largest domain is $(-\infty, -2) \cup [2, \infty)$



(c)

$$f(x) = \ln(3x^2 - 4x + 5)$$

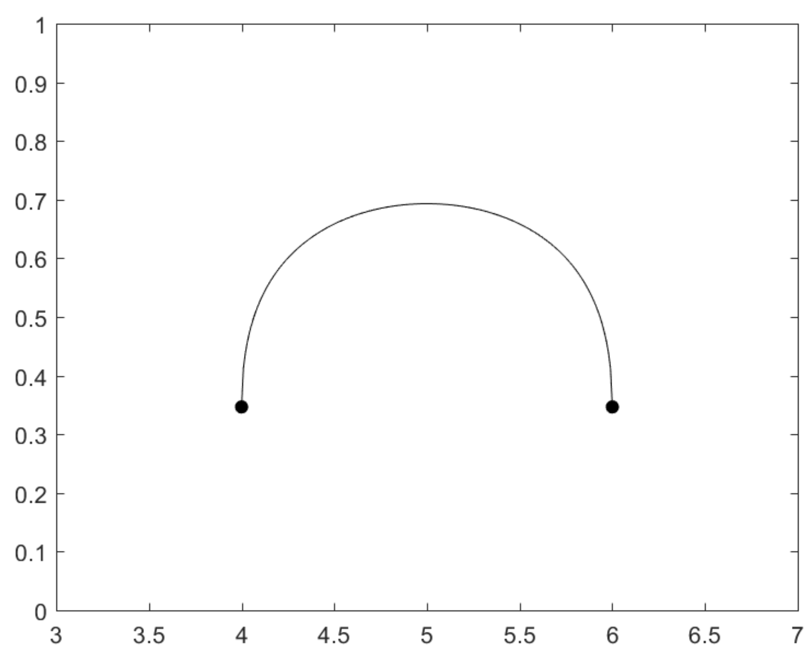
It implies the condition $3x^2 - 4x + 5 > 0$.
Note that $\Delta = (-4)^2 - 4 \cdot 3 \cdot 5 = -44 < 0$, so the equation has no real roots.
Then $3x^2 - 4x + 5 > 0$ for any x.
Hence the largest domain is $(-\infty, \infty)$



(d)

$$f(x) = \ln(\sqrt{x-4} + \sqrt{6-x})$$

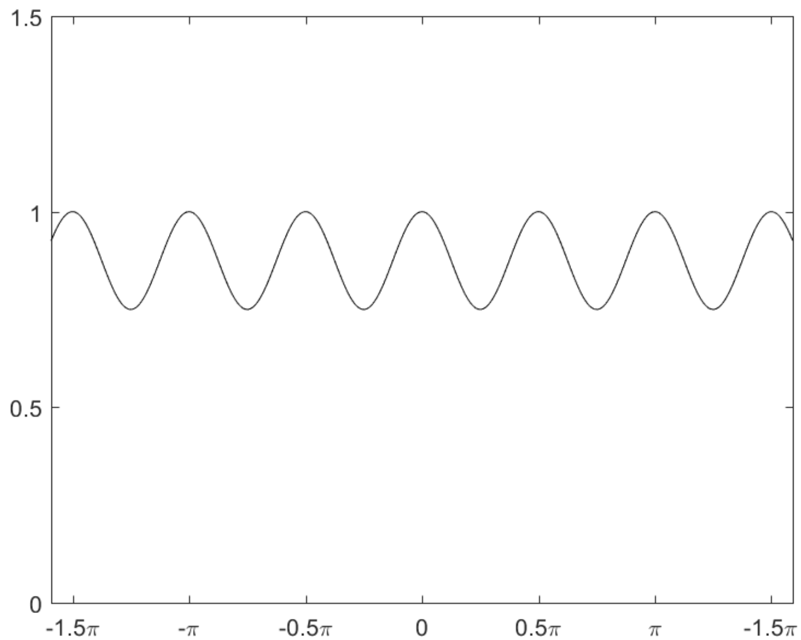
It implies three conditions $x - 4 \geq 0$, $6 - x \geq 0$, and $\sqrt{x-4} + \sqrt{6-x} > 0$.
We get $4 \leq x \leq 6$ from the first two conditions.
For the third condition, note that $\sqrt{x-4} \geq 0$ and $\sqrt{6-x} \geq 0$, and they cannot be 0 simultaneously, so any number satisfying $4 \leq x \leq 6$ works.
Hence the largest domain is $[4, 6]$



(e)

$$f(x) = \sin^2 x + \cos^4 x$$

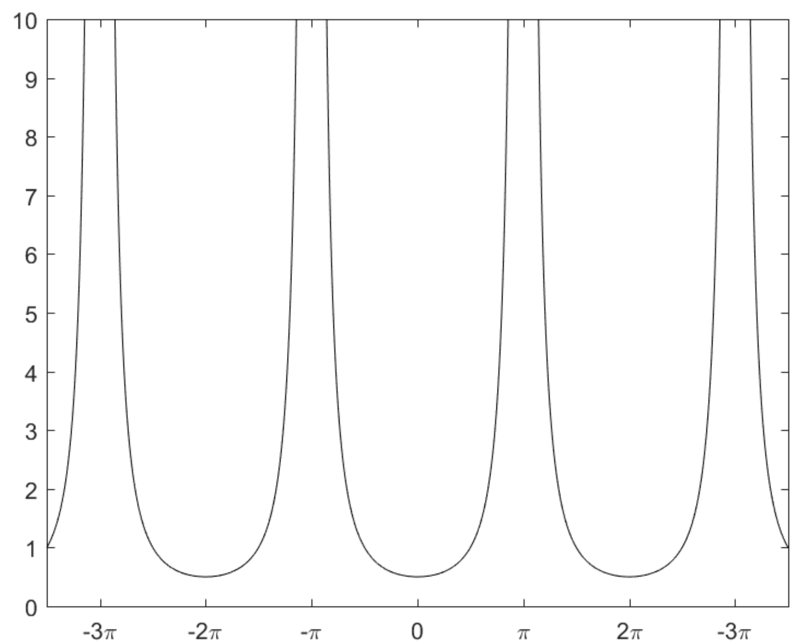
Note that $\sin x$ and $\cos x$ do not impose any conditions on domain.
Hence the largest domain is $(-\infty, \infty)$



(f)

$$f(x) = \frac{1}{1 + \cos x}$$

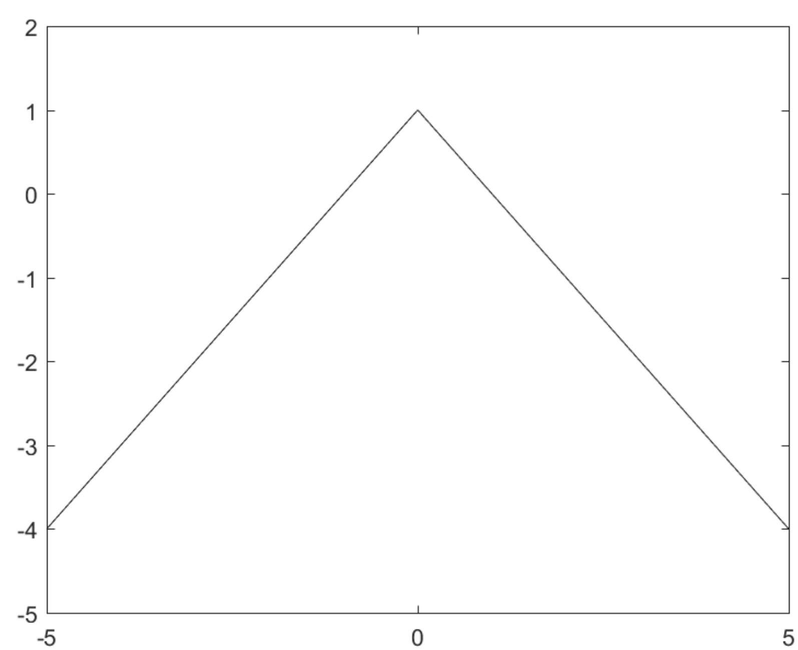
It implies the condition $\cos x \neq -1$.
Then $x \neq \pi + 2n\pi$, where n is any integer.
To write the largest domain in disjoint interval, it involves infinitely many intervals of the form $((2n+1)\pi, (2n+3)\pi)$
We can write it as $\bigcup_{n \in \mathbb{Z}} ((2n+1)\pi, (2n+3)\pi)$



(g)

$$f(x) = 1 - |x|$$

Note that $|x|$ do not impose any conditions on domain.
Hence the largest domain is $(-\infty, \infty)$



8. Determine whether the given function, f, is injective, surjective, bijective, or none of these. Explain clearly.

(a) $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 2x - 1$.

(b) $f : \{x|\ x \neq 1\} \to \mathbb{R}$, where $f(x) = \dfrac{x^2-1}{x-1}$.

(c) $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \sqrt[3]{x}$.

(d) $f : [-1, 1] \to [0, 4)$, where $f(x) = x^2$.

Solutions:

(a) For any $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$, we have $f(x_1) = 2x_1 - 1 \neq f(x_2) = 2x_2 - 1$. Then $f(x)$ is injective.
For any real number $y \in \mathbb{R}$, there exists $x = \frac{y+1}{2} \in \mathbb{R}$ such that $f(x) = y$. Then $f(x)$ is surjective.
Thus, $f(x)$ is bijective since it is both injective and surjective.

(b) $f(x) = x + 1$, for $x \in (-\infty, 1) \cup (1, +\infty)$.
For any $x_1, x_2 \in (-\infty, 1) \cup (1, +\infty)$ with $x_1 \neq x_2$, we have $f(x_1) = x_1 + 1 \neq f(x_2) = x_2 + 1$. Then $f(x)$ is injective.
For real number $y = 2$, there exists no $x \in (-\infty, 1) \cup (1, +\infty)$ such that $f(x) = y$. For otherwise, $x^2 - 1 = 2(x - 1) \implies (x - 1)^2 = 0 \implies x = 1$, which is a contradiction. So $f(x)$ is not surjective.
Thus, $f(x)$ is not bijective.

(c) For any $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$, we have $f(x_1) = \sqrt[3]{x_1} \neq f(x_2) = \sqrt[3]{x_2}$. Then $f(x)$ is injective.
For any real number $y \in \mathbb{R}$, there exists $x = y^3 \in \mathbb{R}$ such that $f(x) = y$. Then $f(x)$ is surjective.
Thus, $f(x)$ is bijective since it is both injective and surjective.

(d) For $x_1 = -x_2$, $x_1, x_2 \in [-1, 1]$, we have $f(x_1) = f(x_2)$. Then $f(x)$ is not injective.
For $y < 0$, there exists no $x \in [-1, 1]$ such that $f(x) = y$. Then, $f(x)$ is not surjective.
Thus, $f(x)$ is not bijective.

9. Determine whether the given function, f, is increasing, strictly increasing, decreasing, strictly decreasing, bounded, bounded above, or bounded below.

(a) $f : [0, +\infty) \to \mathbb{R}$, where $f(x) = \dfrac{x}{x+1}$.

(b) $f : \mathbb{R}^+ \to \mathbb{R}$, where $f(x) = \dfrac{1}{x}$.

Solutions:

(a)

$$f(x) = 1 - \frac{1}{x+1}$$

For any x, y with $x < y$ and $x, y \in [0, +\infty)$, we have $f(x) < f(y)$. Then $f(x)$ is strictly increasing.
For $x \in [0, +\infty)$, $0 = f(0) \leq f(x) \leq \lim_{x \to +\infty} f(x) = 1$. Then $f(x)$ is bounded.

(b) For any x, y with $x < y$ and $x, y \in (0, +\infty)$, we have $f(x) > f(y)$. Then $f(x)$ is strictly decreasing.
Clearly, $f(x) = 1/x > 0$ for any $x \in \mathbb{R}^+$. So f is bounded below by 0. On the other hand, f is not bounded above. Otherwise, if $f(x) \leq M$ for any $x \in \mathbb{R}^+$, then, in particular, $M + 1 = f(1/(M+1)) \leq M$, which is a contradiction.

10. Find whether the function is even, odd or neither:

(a) (Optional) $f(x) = x^2 - |x|$

(b) $f(x) = \log_2 \left(x + \sqrt{x^2+1}\right)$

(c) (Optional) $f(x) = x \left(\frac{a^x - 1}{a^x + 1}\right)$

(d) $f(x) = \sin x + \cos x$

Solutions:

(a)

$$f(-x) = x^2 - |x| = f(x)$$

Thus, $f(x)$ is even.

(b)

$$\begin{aligned} f(-x) &= \log_2 \left(-x + \sqrt{x^2+1}\right) \\ &= \log_2 \left((-x + \sqrt{x^2+1}) \cdot \frac{x + \sqrt{x^2+1}}{x + \sqrt{x^2+1}} \right) \\ &= \log_2 \left(\frac{1}{x + \sqrt{x^2+1}} \right) \\ &= -f(x) \end{aligned}$$

Thus, $f(x)$ is odd.

(c)

$$\begin{aligned} f(-x) &= -x(\frac{a^{-x} - 1}{a^{-x} + 1}) \\ &= x(\frac{a^x - 1}{a^x + 1}) \\ &= f(x) \end{aligned}$$

Thus, $f(x)$ is even.

(d)

$$\begin{aligned} f(-x) &= \sin(-x) + \cos(-x) \\ &= -\sin x + \cos x \end{aligned}$$

$f(x)$ is neither even nor odd since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$.

11. Without using l'Hôpital's rule, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

(a) $\lim_{x \to 3} \dfrac{x^3 - 3x^2 + 5x - 15}{x^2 - x - 12}$.

(b) (Optional) $\lim_{x \to 1/2} \dfrac{1 - 32x^5}{1 - 8x^3}$.

(c) (Optional) $\lim_{x \to 1} \dfrac{x - \sqrt{2 - x^2}}{2x - \sqrt{2 + 2x^2}}$.

(d) $\lim_{x \to 1} \dfrac{\sqrt{x^2 + 8} - \sqrt{10 - x^2}}{\sqrt{x^2 + 3} - \sqrt{5 - x^2}}$.

(e) (Optional) $\lim_{x \to 1} \left(\dfrac{2}{1 - x^2} + \dfrac{1}{x - 1} \right)$.

(f) $\lim_{x \to a} \left(\dfrac{2a}{x^2 - a^2} - \dfrac{1}{x - a} \right)$.

(g) $\lim_{x \to a} \left(\dfrac{x^m - a^m}{x^n - a^n} \right)$.

(h) $\lim_{x \to 1} \left(\dfrac{x - 1}{x^{1/4} - 1} \right)$.

(i) (Optional) $\lim_{x \to 0} \left(\dfrac{x^{7/10} + 3x^{4/3} + 2x}{x^{1/3} + 4x^{2/3} + 2x^{1/5}} \right)$.

Solutions:

(a)

$$\begin{aligned} &\lim_{x \to 3} \frac{x^3 - 3x^2 + 5x - 15}{x^2 - x - 12} \\ =& \frac{3^3 - 3(3^2) + 5(3) - 15}{3^2 - 3 - 12} \\ =& 0 \end{aligned}$$

(b)

$$\begin{aligned}
&\lim_{x\to 1/2}\frac{1-32x^5}{1-8x^3}\\
&=\lim_{x\to 1/2}\frac{(1-2x)(1+2x+4x^2+8x^3+16x^4)}{(1-2x)(1+2x+4x^2)}\\
&=\lim_{x\to 1/2}\frac{1+2x+4x^2+8x^3+16x^4}{1+2x+4x^2}\\
&=\frac{1+2(\frac{1}{2})+4(\frac{1}{2})^2+8(\frac{1}{2})^3+16(\frac{1}{2})^4}{1+2(\frac{1}{2})+4(\frac{1}{2})^2}\\
&=\frac{5}{3}
\end{aligned}$$

(c)

$$\begin{aligned}
&\lim_{x\to 1}\frac{x-\sqrt{2-x^2}}{2x-\sqrt{2+2x^2}}\\
&=\lim_{x\to 1}\frac{x-\sqrt{2-x^2}}{2x-\sqrt{2+2x^2}}\cdot\frac{x+\sqrt{2-x^2}}{2x+\sqrt{2+2x^2}}\cdot\frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}}\\
&=\lim_{x\to 1}\frac{x^2-(2-x^2)}{4x^2-(2+2x^2)}\cdot\frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}}\\
&=\lim_{x\to 1}\frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}}\\
&=\frac{2(1)+\sqrt{2+2(1)^2}}{1+\sqrt{2-1^2}}\\
&=2
\end{aligned}$$

(d)

$$\begin{aligned}
&\lim_{x\to 1}\frac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\\
&=\lim_{x\to 1}\frac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\cdot\frac{\sqrt{x^2+8}+\sqrt{10-x^2}}{\sqrt{x^2+3}+\sqrt{5-x^2}}\cdot\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\\
&=\lim_{x\to 1}\frac{x^2+8-(10-x^2)}{x^2+3-(5-x^2)}\cdot\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\\
&=\lim_{x\to 1}\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\\
&=\frac{\sqrt{1^2+3}+\sqrt{5-1^2}}{\sqrt{1^2+8}+\sqrt{10-1^2}}\\
&=\frac{2}{3}
\end{aligned}$$

(e)

$$\begin{aligned}
&\lim_{x\to 1} \frac{2}{1-x^2} + \frac{1}{x-1} \\
=&\lim_{x\to 1} \frac{2-(1+x)}{(1-x)(1+x)} \\
=&\lim_{x\to 1} \frac{1}{1+x} \\
=&\frac{1}{1+1} \\
=&\frac{1}{2}
\end{aligned}$$

(f)

$$\begin{aligned}
&\lim_{x\to a} \frac{2a}{x^2-a^2} - \frac{1}{x-a} \\
=&\lim_{x\to a} \frac{2a-(x+a)}{(x-a)(x+a)} \\
=&\lim_{x\to a} \frac{-1}{x+a}
\end{aligned}$$

(Case 1) If $a \neq 0$,

$$\begin{aligned}
&\lim_{x\to a} \frac{-1}{x+a} \\
=&\frac{-1}{a+a} \\
=&-\frac{1}{2a}
\end{aligned}$$

(Case 2) If $a = 0$, the limit does not exist since

$$\lim_{x\to a^-} \frac{-1}{x+a} = \lim_{x\to 0^-} \frac{-1}{x} = +\infty$$

while

$$\lim_{x\to a^+} \frac{-1}{x+a} = \lim_{x\to 0^+} \frac{-1}{x} = -\infty$$

(g)

$$\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n}$$

(Case 1) Suppose $a \neq 0$.

$$\begin{aligned}
&\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} \\
=&\lim_{x\to a} \frac{mx^{m-1}}{nx^{n-1}} \quad \text{(l'Hôpital's rule)} \\
=&\frac{m}{n} a^{m-n}
\end{aligned}$$

Alternative answer without using l'Hôpital's rule:

If $m = 0$, then

$$\frac{x^m - a^m}{x - a} = \frac{1 - 1}{x - a} = 0.$$

If $m > 0$, then

$$\lim_{x \to a} \frac{x^m - a^m}{x - a} = \lim_{x \to a} \sum_{k=0}^{m-1} x^k a^{m-1-k} = \sum_{k=0}^{m-1} a^{m-1} = m a^{m-1}.$$

If $m < 0$, then by the above limit,

$$\lim_{x \to a} \frac{x^m - a^m}{x - a} = \lim_{x \to a} -x^m a^m \cdot \frac{x^{-m} - a^{-m}}{x - a} = -a^{2m}(-m)a^{-m-1} = m a^{m-1}.$$

Hence, if $n \neq 0$, we have

$$\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \lim_{x \to a} \frac{x^m - a^m}{x - a} \cdot \frac{x - a}{x^n - a^n} = \frac{m}{n} a^{m-n}.$$

(Case 2) If $a = 0$ and $m = n$,

$$\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = 1$$

(Case 3) If $a = 0$ and $m > n$,

$$\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \lim_{x \to 0} x^{m-n} = 0$$

(Case 4) If $a = 0$ and $m < n$, the limit does not exist since

$$\lim_{x \to a^+} \frac{x^m - a^m}{x^n - a^n} = \lim_{x \to 0^+} \frac{1}{x^{n-m}} = +\infty,$$

while

$$\lim_{x \to a^-} \frac{x^m - a^m}{x^n - a^n} = \lim_{x \to 0^-} \frac{1}{x^{n-m}} = -\infty.$$

(h)

$$\begin{aligned}
&\lim_{x \to 1} \frac{x - 1}{x^{1/4} - 1} \\
=& \lim_{x \to 1} \frac{(x^{1/4} - 1)(x^{1/4} + 1)(x^{1/2} + 1)}{x^{1/4} - 1} \\
=& \lim_{x \to 1} (x^{1/4} + 1)(x^{1/2} + 1) \\
=& (1 + 1)(1 + 1) \\
=& 4
\end{aligned}$$

(i)

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\ln(x+1)}\\
=&\lim_{x\to 0}\frac{(2\sqrt{x+1})^{-1}}{(x+1)^{-1}} \quad \text{(l'Hôpital's rule)}\\
=&\frac{\sqrt{0+1}}{2}\\
=&\frac{1}{2}
\end{aligned}$$

See 11(h) for an answer without using l'Hôpital's rule.

(j)

$$\begin{aligned}
&\lim_{x\to 0}\frac{x^{7/10}+3x^{4/3}+2x}{x^{1/3}+4x^{2/3}+2x^{1/5}}\\
=&\lim_{x\to 0}\frac{x^{1/2}+3x^{17/15}+2x^{4/5}}{x^{2/15}+4x^{7/15}+2}\\
=&\frac{0+0+0}{0+0+2}\\
=&0
\end{aligned}$$

12. Without using l'Hôpital's rule, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

(a) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x^4+1}-\sqrt{x^4-1}}{x}$.

(b) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{3x^2-1}-\sqrt{2x^2+1}}{4x+3}$.

(c) $\displaystyle\lim_{x\to\pi/2}\left(\frac{1-\sin^3 x}{1-\sin^2 x}\right)$.

(d) $\displaystyle\lim_{x\to\pi/4}\left(\frac{\sin 2x-(1+\cos(2x))}{\cos x-\sin x}\right)$.

(e) $\displaystyle\lim_{x\to\pi/4}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}$.

(f) $\displaystyle\lim_{x\to 0}\frac{\sin 7x-\sin x}{\sin 6x}$.

(g) $\displaystyle\lim_{x\to 0}\left(\frac{1+x}{1-x}\right)^{1/x}$.

(h) $\displaystyle\lim_{x\to 0}\left(\frac{\sqrt{x+1}-1}{\ln(1+x)}\right)$.

(i) $\lim_{x\to 0}\left(\frac{e^{ax}-e^{a}}{x}\right)$ where a is a constant.

Solutions:

(a)

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{x^4+1}-\sqrt{x^4-1}}{x} &= \lim_{x\to\infty}\frac{(\sqrt{x^4+1}-\sqrt{x^4-1})(\sqrt{x^4+1}+\sqrt{x^4-1})}{x(\sqrt{x^4+1}+\sqrt{x^4-1})}\\ &= \lim_{x\to\infty}\frac{2}{x(\sqrt{x^4+1}+\sqrt{x^4-1})}\\ &= 0\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{3x^2-1}-\sqrt{2x^2+1}}{4x+3} &= \lim_{x\to\infty}\frac{\sqrt{3-\frac{1}{x^2}}-\sqrt{2+\frac{1}{x^2}}}{4+\frac{3}{x}}\\ &= \frac{\sqrt{3}-\sqrt{2}}{4}\end{aligned}$$

(c)

$$x^3-1=(x-1)(x^2+x+1)$$

$$\begin{aligned}\lim_{x\to\pi/2}\left(\frac{1-\sin^3 x}{1-\sin^2 x}\right) &= \lim_{x\to\pi/2}\frac{(1-\sin x)(1+\sin x+\sin x)}{(1-\sin x)(1+\sin x)}\\ &= \lim_{x\to\pi/2}\frac{(1+\sin x+\sin x)}{(1+\sin x)}\\ &= \lim_{x\to\pi/2}\frac{1+2\sin x}{1+\sin x}\\ &= \frac{3}{2}\end{aligned}$$

(d)

$$1+2\cos 2x = 1+\cos^2 x-\sin^2 x$$

$$\sin 2x = 2\sin x\cos x$$

$$\begin{aligned}\lim_{x\to\pi/4}\left(\frac{\sin\,2x-(1+\cos\,2x)}{\cos\,x-\sin\,x}\right) &= \lim_{x\to\pi/4}\frac{2\cos x(\sin x-\cos x)}{\cos x-\sin x}\\ &= \lim_{x\to\pi/4}-2\cos x\\ &= -\sqrt{2}\end{aligned}$$

(e)

$$a\cos x+b\sin x=\sqrt{a^2+b^2}\sin(x+\tan^{-1}\frac{a}{b}),$$

for $b\neq 0$ and $-\frac{\pi}{2}<\tan^{-1}\frac{a}{b}<\frac{\pi}{2}$.

$$1-\cos x = 2\sin^2(\frac{x}{2})$$

Thus, we have

$$\begin{aligned}\cos x + \sin x &= \sqrt{2}\sin(x + \frac{\pi}{4}) \\ &= \sqrt{2}\cos(x - \frac{\pi}{4})\end{aligned}$$

$$\begin{aligned}\lim_{x\to\pi/4} \frac{\sqrt{2} - \cos\ x - \sin\ x}{(4x-\pi)^2} &= \lim_{x\to\pi/4} \frac{\sqrt{2} - \sqrt{2}\cos(x - \frac{\pi}{4})}{(4x-\pi)^2} \\ &= \lim_{x\to\pi/4} \frac{\sqrt{2}}{16} \times \frac{1-\cos(x-\frac{\pi}{4})}{(x-\frac{\pi}{4})^2} \\ &= \frac{\sqrt{2}}{16} \lim_{x\to\pi/4} \frac{2\sin^2(\frac{x}{2}-\frac{\pi}{8})}{4(\frac{x}{2}-\frac{\pi}{8})^2} \\ &= \frac{\sqrt{2}}{32} \lim_{x\to\pi/4} \frac{\sin^2(\frac{x}{2}-\frac{\pi}{8})}{(\frac{x}{2}-\frac{\pi}{8})^2} \\ &= \frac{\sqrt{2}}{32} \lim_{x\to\pi/4} (\frac{\sin(\frac{x}{2}-\frac{\pi}{8})}{\frac{x}{2}-\frac{\pi}{8}})^2 \\ &= \frac{\sqrt{2}}{32}\end{aligned}$$

(f)

$$\begin{aligned}\lim_{x\to 0} \frac{\sin\ 7x - \sin\ x}{\sin\ 6x} &= \lim_{x\to 0} \frac{\sin 6x \cos x + \cos 6x \sin x - \sin x}{\sin 6x} \\ &= \lim_{x\to 0}(\cos x + \frac{\sin x(\cos 6x - 1)}{\sin 6x}) \\ &= \lim_{x\to 0} \cos x + \lim_{x\to 0} \frac{\sin x(-2\sin^2 3x)}{2\sin 3x \cos 3x} \\ &= \lim_{x\to 0} \cos x - \lim_{x\to 0} \sin x \tan 3x \\ &= 1 + 0 = 1\end{aligned}$$

(g)

$$\begin{aligned}\lim_{x\to 0} \left(\frac{1+x}{1-x}\right)^{1/x} &= \lim_{x\to 0} (1+x)^{1/x} (1-x)^{1/(-x)} \\ &= e \cdot e \\ &= e^2.\end{aligned}$$

(h)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\sqrt{x+1}-1}{\ln(1+x)}\right) &= \lim_{x\to 0}\frac{x}{\ln(x+1)}\cdot\frac{\sqrt{x+1}-1}{x}\\ &= \lim_{x\to 0}\frac{x}{\ln(x+1)}\cdot\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1))}\\ &= \lim_{x\to 0}\frac{x}{\ln(x+1)}\cdot\frac{1}{\sqrt{x+1}+1}\\ &= \lim_{x\to 0}\frac{x}{\ln(x+1)}\cdot\frac{1}{(\sqrt{x+1}+1)}\\ &= \frac{1}{2}\end{aligned}$$

(i) First assume $a \neq 0$.

$$\begin{aligned}\lim_{x\to 0}\left(\frac{e^{ax}-e^{a}}{x}\right) &= a\lim_{x\to 0}\frac{e^{ax}-1+1-e^{a}}{ax}\\ &= a(\lim_{x\to 0}(\frac{e^{ax}-1}{ax}+\frac{1-e^{a}}{ax}))\end{aligned}$$

Now $\lim\limits_{x\to 0}\dfrac{e^{ax}-1}{ax} = 1$ while

$$\lim_{x\to 0^+}\frac{1-e^{a}}{x} = \begin{cases}+\infty & \text{if } a<0\\ -\infty & \text{if } a>0\end{cases} \quad\text{and}\quad \lim_{x\to 0^-}\frac{1-e^{a}}{x} = \begin{cases}-\infty & \text{if } a<0\\ +\infty & \text{if } a>0\end{cases}$$

Thus

$$\lim_{x\to 0^+}\left(\frac{e^{ax}-e^{a}}{x}\right) = \begin{cases}+\infty & \text{if } a<0\\ 0 & \text{if } a=0\\ -\infty & \text{if } a>0\end{cases} \quad\text{and}\quad \lim_{x\to 0^-}\left(\frac{e^{ax}-e^{a}}{x}\right) = \begin{cases}-\infty & \text{if } a<0\\ 0 & \text{if } a=0\\ +\infty & \text{if } a>0\end{cases}.$$

13. Evaluate the following limits.

(a) $\lim\limits_{x\to 0^-} x|\sin\frac{1}{x}|$

(b) $\lim\limits_{x\to +\infty}\dfrac{\sin(\tan x)+\tan(\sin x)}{x+1}$

Solutions:

(a)

$$\lim_{x\to 0^-} x\left|\sin\frac{1}{x}\right|$$

Note that $0 \leq \left|\sin\dfrac{1}{x}\right| \leq 1$

Then $-x \leq x\left|\sin\dfrac{1}{x}\right| \leq x$

Since $\lim_{x\to 0} -x = 0$ and $\lim_{x\to 0} x = 0$,

by sandwich theorem, $\lim_{x\to 0} x\left|\sin\frac{1}{x}\right| = 0$

Then $\lim_{x\to 0^-} x\left|\sin\frac{1}{x}\right| = 0$

(b)

$$\lim_{x\to +\infty}\frac{\sin\tan x + \tan\sin x}{x+1}$$

Note that $-1 \le \sin x \le 1$

Then $-\tan 1 \le \tan\sin x \le \tan 1$

$-\frac{1+\tan 1}{x+1} \le \frac{\sin\tan x + \tan\sin x}{x+1} \le \frac{1+\tan 1}{x+1}$ for $x > 0$

Since $\lim_{x\to +\infty} -\frac{1+\tan 1}{x+1} = 0$ and $\lim_{x\to +\infty}\frac{1+\tan 1}{x+1} = 0$,

by sandwich theorem, $\lim_{x\to +\infty}\frac{\sin\tan x + \tan\sin x}{x+1} = 0$

14. Evaluate the following limits.

(a) $\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}$

(b) $\lim_{x\to 0}\frac{\tan^2 x}{\sin(x^2)}$

(c) $\lim_{x\to 0}\frac{\sin^2 x}{1-\sqrt{\cos x}}$

Solutions:

(a)

$$\begin{aligned}
&\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}\\
&= \lim_{x\to 0}\frac{1-\cos x}{\sin^2 x\cos x}\\
&= \lim_{x\to 0}\frac{1-\cos x}{(1-\cos x)(1+\cos x)\cos x}\\
&= \lim_{x\to 0}\frac{1}{(1+\cos x)\cos x}\\
&= \frac{1}{(1+1)(1)}\\
&= \frac{1}{2}
\end{aligned}$$

(b)

$$
\begin{aligned}
&\lim_{x\to 0}\frac{\tan^2 x}{\sin(x^2)}\\
&=\lim_{x\to 0}\frac{\dfrac{\tan^2 x}{x^2}}{\dfrac{\sin(x^2)}{x^2}}\\
&=\lim_{x\to 0}\frac{\dfrac{\sin x}{x}\dfrac{\sin x}{x}\dfrac{1}{\cos^2 x}}{\dfrac{\sin(x^2)}{x^2}}\\
&=\frac{(1)(1)\left(\dfrac{1}{1}\right)}{1}\\
&=1
\end{aligned}
$$

(c)

$$
\begin{aligned}
&\lim_{x\to 0}\frac{\sin^2 x}{1-\sqrt{\cos x}}\\
&=\lim_{x\to 0}\frac{\sin^2 x}{1-\sqrt{\cos x}}\cdot\frac{1+\sqrt{\cos x}}{1+\sqrt{\cos x}}\cdot\frac{1+\cos x}{1+\cos x}\\
&=\lim_{x\to 0}\frac{\sin^2 x}{1-\cos^2 x}(1+\sqrt{\cos x})(1+\cos x)\\
&=\lim_{x\to 0}(1+\sqrt{\cos x})(1+\cos x)\\
&=(1+1)(1+1)\\
&=4
\end{aligned}
$$