Recall: Thm (Volume comparison):

If (M,g) is a complete Riemannian mfld w/ $Ric(g) \geq (n-1)k$ for some $k \in \mathbb{R}$, then $\forall p \in M$

- $\dfrac{Vol(B(p,r))}{\underbrace{Vol_k(B_k(r))}}$ is non-increasing in $r>0$.

  volume of Ball of Radius r in spaceform with $K \equiv k$.

- And $vol(p,r) = vol_k(r)$ for some $r>0$ iff

$$B(p,r) \overset{\text{iso}}{\cong} B_k(r).$$

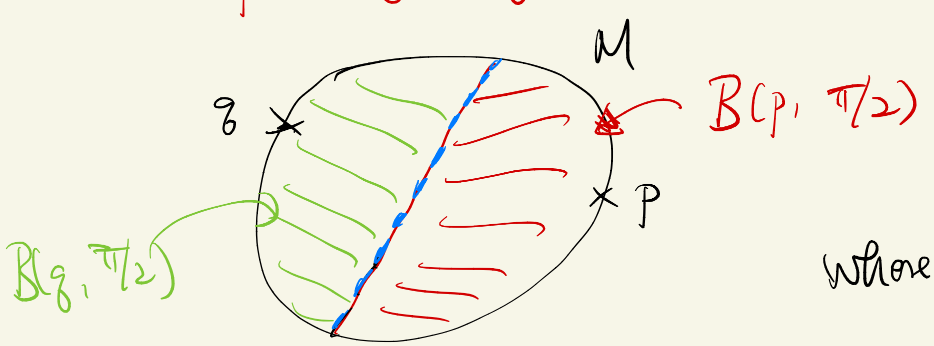
Thm (Cheng) If (M,g) is complete with $Ri \geq (n-1)k$, for some $k>0$, then $\boxed{diam(M,g) \leq \frac{\pi}{\sqrt{k}}}$. And Equality holds iff $(M,g) \cong (S^n, g_{std})$

by Meyers thm from 2nd variational formula

---

pf: Assume $k=1$ by scaling.

If $\underset{\substack{\| \\ \sup\{d(x,y) \mid x,y \in M\}}}{diam(M,g)} = \pi \Rightarrow \exists\, p,q \in M$ s.t. $d(p,q) = \pi$.



By volume comparison,

$$V(p,\pi) \leq V(p,\tfrac{\pi}{2}) \cdot \frac{\bar{V}(\pi)}{\bar{V}(\pi/2)}$$

where $\bar{V}(r)$ = volume of $\bar{B}(r)$ in sphere.

$\because$ $(\bar{M}, \bar{g}) = S^n$ with std metric

$\therefore$ $\bar{V}(\pi) = 2\,\bar{V}(\pi/2)$

$$\Rightarrow \quad V(p, \pi) \leq 2\, V(p, \pi/2)$$

similarly, $\Rightarrow V(q, \pi) \leq 2\, V(q, \pi/2)$

$$V(p, \pi) + V(q, \pi) \leq 2\left(V(p, \pi/2) + V(q, \pi/2)\right)$$

$\Big\|$ diam $= \pi$ $\qquad \leq 2\,\mathrm{Vol}(M)$

$2\,\mathrm{Vol}(M)$

since $B(p, \pi/2) \cap B(q, \pi/2) = \phi$.

otherwise $d(p,x) + d(q,x) < \pi$

$\qquad\qquad \rotatebox{90}{$\leq$}$

$\qquad d(p,q) = \pi$.

$\Rightarrow$ All inequalities above are equalities !!

(tracing the proof)

$\Rightarrow$ $(M, g) \cong (S^n, g_{std})$ #.

Thm (Laplacian comparison)

If (M,g) is complete Riemannian with $Ric \geq (n-1)k$. for some $k \in \mathbb{R}$, then for a distance fcn $r(x) = d(x,p)$ for $p \in M$.

$$\Delta r(x) \leq \begin{cases} (n-1)\sqrt{k}\,\cot(\sqrt{k}\,r) & \text{if } k>0 \\ \frac{n-1}{r} & \text{if } k=0 \\ (n-1)\sqrt{|k|}\,\coth(\sqrt{|k|}\,r) & \text{if } k<0. \end{cases}$$

① in the sense of distribution

② in the classical sense whenever $r(\cdot)$ is smooth (away from cut pt)

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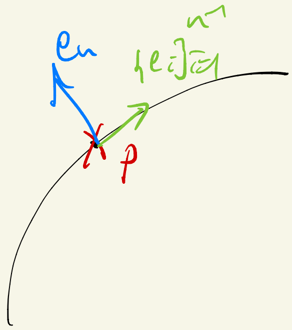
pf: (General discussion) Let $f \in C^\infty(M)$ s.t.



$\{f = \text{const}\} = \Sigma$

Goal: Compute Δf using Σ.

$$\Delta f = \operatorname{tr}_g(\nabla^2 f) = g^{ij}\nabla_i\nabla_j f \quad \text{(locally)}$$

$$\Delta f = \sum_{i=1}^{n-1} \nabla_i\nabla_i f + \nabla_n\nabla_n f \qquad \text{(using the above coordinate/frame)}$$

$$= \sum_{i=1}^{n-1} \left(\partial_i\partial_i f - \Gamma_{ii}^k f_k\right) + \nabla_n\nabla_n f$$

$$= \sum_{i=1}^{n-1} \left(-\Gamma_{ii}^n\right) f_n + \nabla_n\nabla_n f \qquad \text{and } (\partial_i f = 0 \ \forall i \neq n)$$

$$= \vec{H}(f) + (\nabla^2 f)(e_n, e_n)$$

Normal vector.

---

take $f = r(x)$ whenever $r(\cdot)$ is smooth

$$\Rightarrow \quad \vec{H} = H\nabla r \quad \Rightarrow \quad \Delta r(x) = H(x) \qquad (\because (\nabla r)(r) = \langle \nabla r, \nabla r\rangle = 1)$$

Recall: $H(x) \le \overline{H}(r(x))$ from volume comparison thm.

By the expression (explicit) of $\overline{H}$, Done whenever $r(\cdot)$ is smooth!!

Claim: the inequalities hold in distributional sense.

$\Updownarrow$

$$\forall \phi \in C_c^\infty(M) \text{ with } \phi \geq 0, \quad \int_M \Delta\phi \cdot r \leq \int_M \bar{H}(r)\,\phi.$$

---

pf:
$$R.H.S = \int_M \phi \cdot \bar{H}(r)$$

co-area formula
$$= \int_0^\infty \left(\int_{C(r)} \phi \cdot \bar{H}(s)\, J(s,\theta)\, d\theta \right) ds$$

(where $C(r) = \{ \theta \in S_pM : \exp_p(s\theta) \text{ is minimizing up to } r \}$)

Fubini thm
$$= \int_{S_pM} \int_0^{R(\theta)} \phi \cdot \bar{H}(s)\, J(s,\theta)\, ds\, d\theta$$

(where $\forall \theta \in S_pM$, $R(\theta)$ is the max s.t. $\exp_p(s\theta)$ is minimizing up to $R(\theta)$.)

volume comp.
$$\geq \int_{S_pM} \left(\int_0^{R(\theta)} \phi \cdot H(s,\theta)\, J(s,\theta)\, ds \right) d\theta$$

first variational formula
$$= \int_{S_pM} \left(\int_0^{R(\theta)} \phi \cdot \partial_s J\, ds \right) d\theta$$

unknown

$$= - \int_{S_pM} \int_0^{R(\theta)} \phi_s \cdot J\, ds\, d\theta + \int_{S_pM} \phi(\theta, s)\, J(\theta, s)\, d\theta$$

$J=0$

$$\geq - \int_{S_pM} \int_0^{R(\theta)} \langle \nabla\phi, \nabla r \rangle \cdot J\, ds\, d\theta$$

integration by part because r is lip.

$$= - \int_M \langle \nabla\phi, \nabla r \rangle = \int_M \Delta\phi \cdot r. \qquad \cancel{\times}$$

Thm (Cheeger - Gromoll thm)

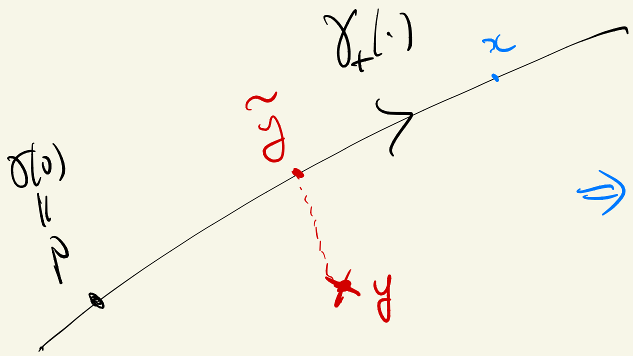
If (M,g) is a complete Riemannian mfd with $Ric \geq 0$ and $\exists$ a normal geodesic $\gamma: (-\infty,\infty) \longrightarrow M$ s.t. $\forall [a,b] \subset \mathbb{R}$, $\gamma|_{[a,b]}$ is minimizing then $M \overset{\text{iso}}{\cong} \mathbb{R} \times N$ with $Ric(N) \geq 0$.

called line

pf: Consider $\gamma_+ : [0,+\infty) \to M$, given by

$\gamma_+(t) = \gamma(t)$. (called Ray)

define $\beta_+(x) = \lim\limits_{t\to\infty} \left(t - d(x, \gamma_+(t))\right)$, $\forall x \in M$.



- if x is on the Ray, then $x = \gamma_+(s)$ for some $s>0$

$\Rightarrow$ $t - d(\gamma(s), \gamma(t)) = t - (t-s) = s$.

$\therefore$ $\beta_+(x) = dist(p, x)$.

- if $y \notin$ Ray.

$t - d(\gamma(t), y) \approx t - d(\gamma(t), \tilde{y}) - d(\tilde{y}, y)$

$\approx \underbrace{d(p, \tilde{y}) - d(\tilde{y}, y)}_{\text{distance-like in some sense}}$

Step 1 claim: $\beta_+(\cdot)$ is well defined, and is lip fcn with lip-constant ≤ 1.

pf: if $t>s$, for $x \in M$.

$$t - d(\gamma_+(t), x) \geq s - d(\gamma_+(s), x) + (t-s) - d(\gamma_+(s), \gamma_+(t))$$
$$= s - d(\gamma_+(s), x) + \cancel{(t-s)} - \cancel{(t-s)}$$

$\Rightarrow \{t - d(\gamma_t(t), x)\}$ is increasing in t, $\forall x \in M$.

$\overset{MCT}{\Longrightarrow}$ $\beta_+(x)$ exists.

Moreover, $\forall x, y \in M$,

$$t - d(\gamma_+(t), x) \geq t - d(\gamma_+(t), y) - d(x,y)$$

$\overset{t\to\infty}{\Longrightarrow}$ $d(x,y) + \beta_+(x) \geq \beta_+(y)$

$\overset{x \leftrightarrow y}{\Longrightarrow}$ $d(x,y) + \beta_+(y) \geq \beta_+(x)$

$\Longrightarrow$ $|\beta_+(x) - \beta_+(y)| \leq d(x,y) \Rightarrow$ 1-lip !!

Step 2: $\Delta\beta_+ \geq 0$ in the distributional sense.

$\Updownarrow$

$\forall \phi \in C_c^\infty(M), \geq 0$ we have $\int_M \beta_+ \cdot \Delta\phi \geq 0$.

pf:

Dominated Convergence thm as $\underbrace{}$ is lip. unif in t.

$$\int_M \Delta\phi \cdot \beta_+ = \lim_{t\to\infty} \int_M \Delta\phi \, (t - d(\gamma_+(t), \cdot))$$

$$\underset{(\phi \in C_c^\infty(M))}{\overset{\text{stoke thm}}{=}} \lim_{t\to\infty} \int_M \Delta\phi \cdot [-d(\gamma_+(t), \cdot)]$$

lap. comp.

$$\geq \lim_{t\to\infty} \int_M -\phi \cdot \left(\frac{n-1}{d(\gamma_+(t),\cdot)} \right)$$

DCT

$$= 0 \quad \text{since} \quad d(\gamma_+(t),\cdot) \to \infty.$$

$\#$

Step 3 : Replace $\gamma_+(t)$ by $\gamma_-(t) = \gamma(-t)$ to obtain

$$\beta_-(x) = \lim_{t\to\infty} \left(t - d(\gamma_-(t), x) \right) \quad \text{s.t.} \quad \begin{cases} \Delta \beta_+ \geq 0 \\ \Delta \beta_- \geq 0. \end{cases}$$

$\Rightarrow$ $\beta_+ + \beta_- = f$ satisfies

① f is Lip

② $\Delta f \geq 0$ (weakly)

③ $f = 0$ along γ.

(since $t - d(\gamma_+(t), \gamma(s)) + t - d(\gamma_-(t), \gamma(s))$

$= t - (t-s) + t - (s+t)$ (if $s>0$)

$= t - t + s + t - s - t = 0.$)

④ f $x \notin$ Ray,

$$2t = d(\gamma_+(t), \gamma_-(t)) \leq d(\gamma_+(t), x) + d(\gamma_-(t), x)$$

$$\Rightarrow \quad \beta_+ + \beta_- = f \leq 0 \quad \text{on} \quad M.$$

$$\text{PDE} \xrightarrow{\text{strong MP}} f \equiv 0 \quad \text{on} \quad M.$$

$$\Rightarrow \boxed{\Delta\beta_+ = -\Delta\beta_- = 0} \quad \text{in distributional sense}$$

together with Lip $\Rightarrow$ β_+, β_- are smooth. by PDE.

$$\Rightarrow \begin{cases} |\nabla \beta_\pm| \leq 1 & \text{in } C^1 \text{ sense} \\ |\nabla \beta_\pm| = 1 & \text{on Ray} \end{cases}$$

Bochner technique :

$$\Delta |\nabla\beta|^2 \xlongequal{\text{Normal coord}} \nabla_i\nabla_i |\beta_j|^2 = \nabla_i (2\beta_j \beta_{ij})$$

$$= 2\beta_{ij}^2 + 2\beta_j \beta_{iji} \qquad \text{(Recall } \beta \text{ is harmonic)}$$

$$\left(\begin{aligned} \beta_{iji} &= \nabla_i\nabla_j\nabla_i\beta \\ &= \nabla_j \cancel{\Delta\beta}^{\,0} - R_{iji}{}^{k}\, \nabla_k \beta = R_j{}^{k}\, \beta_k \end{aligned} \right)$$

$$= 2\beta_{ij}^2 + 2R_j{}^{k}\, \beta_k \beta_j$$

$$= 2|\nabla^2\beta|^2 + 2\,\mathrm{Ric}(\nabla\beta, \nabla\beta) \geq 0$$

$\because$ $|\nabla\beta|^2$ is ① C^∞_{loc} ② sub-harmonic

③ $|\nabla\beta|^2 \leq 1$ ④ $|\nabla\beta| \equiv 1$ on Ray

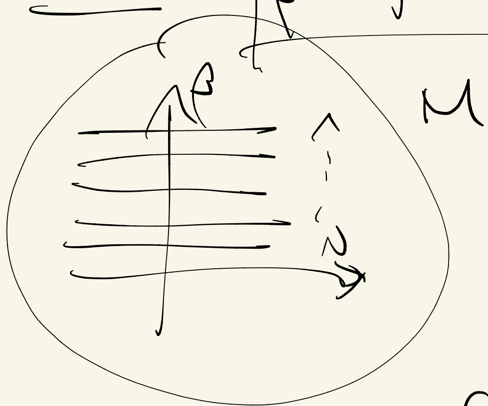
$$\Rightarrow |\nabla\beta| \equiv 1 \text{ on } M.$$

Apply $|\nabla\beta|^2 = 1$ to Bochner formula:

$$\begin{cases} 0 \equiv \Delta|\nabla\beta|^2 = 2|\nabla^2\beta|^2 + 2\,\mathrm{Ric}(\nabla\beta, \nabla\beta) \\ |\nabla\beta| \equiv 1 \end{cases}$$

$$\Rightarrow \begin{cases} \mathrm{Ric}(\nabla\beta, \nabla\beta) = 0 \\ \nabla^2\beta = 0 \end{cases} \text{ on } M \Rightarrow M \text{ splits a } \mathbb{R} \text{ factor}$$

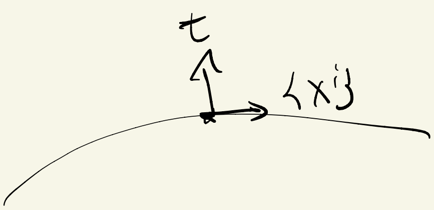
<u>Idea:</u> $\boxed{\beta : \text{parametrizing the level set!!}}$



To Make precise:

Let $N = \beta^{-1}(0)$, which is a smooth sub-mfd by $|\nabla\beta| \neq 0$

choose coordinate around N s.t. $\{x_i\}_{i=1}^{n-1}$ is coordinate of N.



$N = \beta^{-1}(0)$

- Do-able locally.
- Claim that true globally with product structure

dependence (a priori)

- Locally, $g = g_{ij}(x, t)\, dx^i \otimes dx^j + dt^2$ (under this coordinate.)
- Suffices to show that $g_{ij}(x,t) = g_{ij}(x)$

<u>pf:</u> $\partial_t g_{ij} = g(\nabla_{\partial_t}\partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_t}\partial_j)$

$$= g(\nabla_i \partial_t, \partial_j) + g(\partial_i, \nabla_j \partial_t)$$

$$= g(\nabla_i \nabla\beta, \partial_j) + g(\partial_i, \nabla_j \nabla\beta) = 0 \quad \nabla^2\beta = 0$$

$$\therefore \quad M \cong \mathbb{R}\times N \quad \text{with} \quad g = g_N \oplus dt^2 .$$

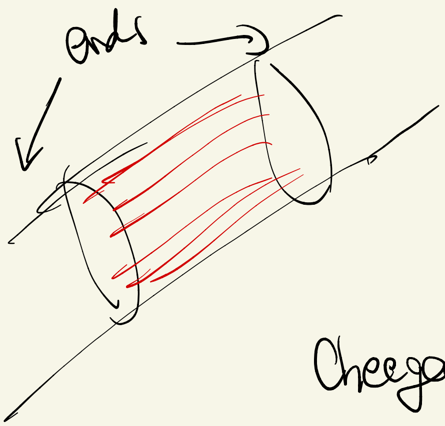
Hence, $Ric(M) \geq 0 \Rightarrow Ric(N) \geq 0$

Conseq: $S^p \times S^1$ CANNOT admit metric with $Ric \equiv 0$ if $p = 2, 3$

pf: Consider the universal cover of $S^p \times S^1$, which is $S^p \times \mathbb{R}$.

From the topology, $S^p \times \mathbb{R}$ contains a line

($\because$ this is dis-connected at ∞)



Suppose it admits a Ricci flat metric on $S^p \times S^1$ and hence on $S^p \times \mathbb{R}$.

Cheeger-Gromoll $\Rightarrow S^p \times \mathbb{R} \overset{\text{iso}}{\cong} N \times \mathbb{R}$

$\therefore N = S^p$ with $Ric(N) = 0$.

If $p = 2, 3 \Rightarrow Rm(N) \equiv 0$

$\Rightarrow S^p$ admits flat metric which is Impossible !!

Application: Thm (2nd part of Torus Rigidity)

If $R_g \geq 0$ on T^n, then $Rm_g = 0$

Pf: Last time, we proved: if $R \geq 0$, then $Ric \equiv 0$.

(proved by non-existence of metrics with $R > 0$)

If g is Ricci flat on T^n,

then lift g to $\mathbb{R}^n$ (which is universal cover of T^n)

then $(\mathbb{R}^n, \tilde{g})$ is Ricci flat

Cheeger-Gromoll $\Rightarrow (\mathbb{R}^n, \tilde{g}) \cong (\mathbb{R}^n, g_{flat})$

$\therefore Rm(g) \equiv 0$ #

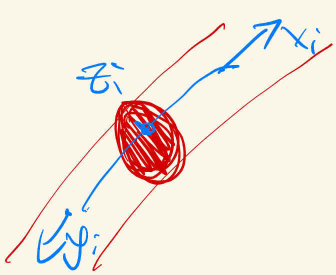
Thm (Lohkamp): Any complete mfd admits metric w/ $Ric < 0$.

☆ In other words, we CANNOT classify mfd with $Ric < 0$.

Corollary: If (M, g) is complete s.t. M admits two ends, $Ric \geq 0$.

then $M \cong N \times \mathbb{R}$ where N is cpt.

Pf:



By assumption, $\exists\, \Omega$ cpt set s.t.

$$M \setminus \Omega = E_1 \sqcup E_2$$

take $\begin{cases} x_i \in E_1 & \text{s.t. } x_i \to +\infty \\ y_i \in E_2 & \text{s.t. } y_i \to +\infty \end{cases}$

Choose a Normal geodesic γ_i passing through x_i, y_i

then $z_i \in \gamma_i \cap \Omega$, WLOG assume $\gamma_i(0) = z_i$

$\Rightarrow \begin{cases} \gamma_i(0) = z_i \to z_\infty \in \Omega \\ \gamma_i'(0) = v_i \to v_\infty \in S_{z_\infty}M \end{cases}$

s.t. $\gamma_i \to \gamma_\infty$ s.t. $\begin{cases} \gamma_\infty(0) = z_\infty \\ \gamma_\infty'(0) = v_\infty . \end{cases}$

then γ_∞ is minimizing $\forall [a,b] \subset \mathbb{R}$.

$\therefore$ M contains a line $\overset{CG}{\Longrightarrow}$ $M = N \times R$.

Case 1: $M = \tilde{N} \times R^k$. where $k > 1$

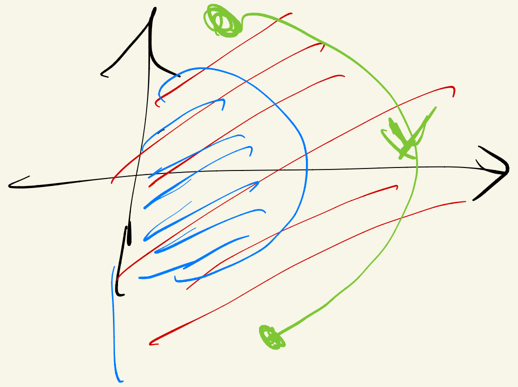
which is impossible because M has two ends

which $\mathbb{R}^k$, $k > 1$ has one end.

Case 2 $M = N \times \mathbb{R}$ where N is Non-cpt without a line

(N is connected at ∞)

picture N "=" $[0, +\infty)$



$\Rightarrow$ M will have one end which is impossible

Case 3: $M = N \times \mathbb{R}$ where N is cpt.

$\not\exists$.