THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2021)
Suggested Solution of Homework 4

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1. (3 points) Show that

(a) $\lim_{x \to 1} \dfrac{x}{1+x} = \dfrac{1}{2}$;

(b) $\lim_{x \to 1} x^2 + 3x = 4$;

(c) $\lim_{x \to 0^+} \sqrt{x} \sin(x^{-1}) = 0$.

Solution:

(a) Let $f(x) = \frac{x}{1+x}$. For any $\epsilon > 0$, we choose $\delta = \min\{\epsilon, 1\}$. Then for $|x-1| < \delta$, we have that

$$\left| f(x) - \frac{1}{2} \right| = \left| \frac{x-1}{2+2x} \right| \leq |x-1| < \epsilon.$$

Therefore, $\lim_{x \to 1} f(x) = \frac{1}{2}$.

(b) Let $f(x) = x^2 + 3x$. For any $\epsilon > 0$, we choose $\delta = \min\left\{\frac{\epsilon}{6}, 1\right\}$. Then for $|x-1| < \delta$, we have that

$$|f(x) - 4| = \left| x^2 + 3x - 4 \right| = |x-1| \, |x+4| \leq 6 \, |x-1| < \epsilon.$$

Therefore, $\lim_{x \to 1} f(x) = 4$.

(c) Let $f(x) = \sqrt{x} \sin(x^{-1})$. For any $\epsilon > 0$, we choose $\delta = \epsilon^2$. Then for $|x-0| < \delta$, we have that

$$|f(x) - 0| = \left| \sqrt{x} \sin(x^{-1}) \right| = \left| \sqrt{x} \right| \left| \sin(x^{-1}) \right| \leq \left| \sqrt{x} \right| < \epsilon.$$

Therefore, $\lim_{x \to 0^+} f(x) = 0$.

2. (2 points) Show that the following limit does not exists.

   (a) $\lim_{x\to 1}(x-1)^{-2}, x > 0$;

   (b) $\lim_{x\to +\infty} \sin x$;

Solution:

(a) Let $f(x) = (x-1)^{-2}$ and $a_n = 1+n^{-1}$. Then $f(a_n) = n^2$. For any $M \in \mathbb{R}$, by the Archimedean property, there exists an $n \in \mathbb{N}$ such that $f(a_n) = n^2 \geq n \geq M$. Hence $f(a_n)$ is an unbounded and divergent sequence. Since $\lim_{n\to\infty} a_n = 1$, we have that $\lim_{x\to 1}(x-1)^{-2}$ does not exist.

(b) Let $f(x) = \sin(x)$, $a_n = 2n\pi$ and $b_n = 2n\pi + \frac{\pi}{2}$. Note that for all $n \in \mathbb{N}$,

$$f(a_n) = \sin(2n\pi) = 0$$

and

$$f(b_n) = \sin(2n\pi + \frac{\pi}{2}) = 1.$$

Then $\lim_{n\to\infty} f(a_n) = 0 \neq 1 = \lim_{n\to\infty} f(b_n)$. Since both a_n and b_n go to $+\infty$ as $n \to \infty$, we have that $\lim_{x\to +\infty} \sin x$ does not exist.

3. (2 points) Define

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{otherwise.} \end{cases}$$

Show that f has limit at $x = 0$ and does not have a limit at $c \in \mathbb{R}$ for all $c \neq 0$.

Solution: Since $0 \in \mathbb{Q}$, we have that $f(0) = 0$. To show f is continuous at 0, it suffices to show that $\lim_{x\to 0} f(x) = 0$. Note that f satisfies that $|f(x)| \leq |x|$ for any $x \in \mathbb{R}$. Let $\epsilon > 0$. We choose $\delta = \epsilon$. Then for $|x - 0| < \delta$, we have that

$$|f(x) - 0| = |f(x)| \leq |x| < \epsilon.$$

Therefore, $\lim_{x\to 0} f(x) = 0$.

For $c \neq 0$, by density of $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$, we can find two sequences a_n and b_n converging to $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $a_n \in \mathbb{Q}$ and $b_n \notin \mathbb{Q}$. Then for all $n \in \mathbb{N}$,

$$f(a_n) = a_n$$

and

$$f(b_n) = 0.$$

It follows that

$$\lim_{n\to\infty} f(a_n) = c \neq 0 = \lim_{n\to\infty} f(b_n).$$

Therefore, f does not have a limit at c.

4. (3 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f has a limit L at $x = 0$.

   (a) Show that $L = 0$;

   (b) Show that f has a limit at every $c \in \mathbb{R}$.

Solution:

(a) For all $x \in \mathbb{R}$, we have that

$$f(2x) = f(x + x) = 2f(x).$$

Since $\lim_{x\to 0} f(x) = L$, we have that

$$L = \lim_{x\to 0} f(x) = \lim_{x\to 0} f(2x) = \lim_{x\to 0} 2f(x) = 2L.$$

It follows that $L = 0$.

(b) Since $\lim_{x\to 0} f(x) = 0$, we have that for any $c \in \mathbb{R}$,

$$\lim_{x\to c} f(x) = \lim_{h\to 0} f(c + h) = \lim_{h\to 0}(f(c) + f(h)) = f(c) + \lim_{h\to 0} f(h) = f(c).$$

Hence, f has a limit at every $c \in \mathbb{R}$.