THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2021)
Suggested Solution of Homework 3

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1. (2 points) If $x_1 = 1$, $x_2 = 2$ and $x_{n+2} = \frac{1}{3}x_{n+1} + \frac{2}{3}x_n$, show that $\{x_n\}_{n=1}^{\infty}$ is convergent and find its limit.

Solution: Note that $|x_{n+2} - x_{n+1}| = \left|\frac{1}{3}x_{n+1} + \frac{2}{3}x_n - x_{n+1}\right| = \frac{2}{3}\left|x_{n+1} - x_n\right|$. Since $|x_2 - x_1| = 1$, we have that $|x_{n+1} - x_n| = (\frac{2}{3})^{n-1}$. Hence for $m, n \in \mathbb{N}$ and $m > n$

$$|x_m - x_n| \leq \sum_{i=n}^{m-1} |x_{i+1} - x_i| \leq \sum_{i=n}^{m-1} (\frac{2}{3})^{i-1} \leq (\frac{2}{3})^{n-1} \sum_{i=1}^{m-n} (\frac{2}{3})^{i-1} \leq 3 \cdot (\frac{2}{3})^{n-1}.$$

For $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that $3 \cdot (\frac{2}{3})^{N-1} < \epsilon$. Then for $m > n \geq N$, $|x_m - x_n| < \epsilon$. Therefore $\{x_n\}_{n=1}^{\infty}$ is Cauchy, thus convergent.

Suppose $\lim_{n\to\infty} x_n = L$. From the identity $x_{n+2} = \frac{1}{3}x_{n+1} + \frac{2}{3}x_n$, we have that for $k \in \mathbb{N}$,

$$\sum_{n=1}^{k+2} x_{n+2} = \sum_{n=1}^{k+2} \frac{1}{3}x_{n+1} + \sum_{n=1}^{k+2} \frac{2}{3}x_n$$

$$x_{k+4} + x_{k+3} + \sum_{n=1}^{k} x_{n+2} = \frac{1}{3}(x_{k+3} + \sum_{n=1}^{k} x_{n+2} + x_2) + \frac{2}{3}(\sum_{n=1}^{k} x_{n+2} + x_1 + x_2)$$

$$x_{k+4} + x_{k+3} = \frac{1}{3}(x_{k+3} + x_2) + \frac{2}{3}(x_1 + x_2)$$

By letting $k \to \infty$, we have that $2L = \frac{1}{3}(L + 2) + \frac{2}{3}(1 + 2)$. Therefore $L = \frac{8}{5}$.

2. (2 points) Prove or disprove the following: Suppose $\sum x_n$ is a convergent series with $x_n > 0$ for all n.

   (a) $\sum x_n^2$ is convergent.

   (b) $\sum \sqrt{x_n}$ is convergent.

Solution:

(a) Since $\sum x_n$ is a convergent series, there exists $N \in \mathbb{N}$ such that $|\sum_{n=N}^{m} x_n| < 1$ for all $m > N$. Since $x_n > 0$ for all $n \in \mathbb{N}$, we have that $x_n < 1$ for $n > N$. It follows that for $n > N$, $0 < x_n^2 < x_n$. By the comparison test, $\sum x_n^2$ converges.

(b) Let $x_n = n^{-2}$. Then $\sum x_n$ converges but $\sum \sqrt{x_n} = \sum \frac{1}{n}$ is not convergent.

3. (3 points) If $f : \mathbb{R} \to \mathbb{R}$ is a function given by $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \notin \mathbb{Q}$, then f is continuous at $x = 0$.

Solution: Since $0 \in \mathbb{Q}$, we have that $f(0) = 0$. To show f is continuous at 0, it suffices to show that $\lim_{x\to 0} f(x) = 0$. Note that f satisfies that $|f(x)| \leq |x|$ for any $x \in \mathbb{R}$. Let $\epsilon > 0$. We choose $\delta = \epsilon$. Then for $|x - 0| < \delta$, we have that

$$|f(x) - 0| = |f(x)| \leq |x| < \epsilon.$$

Therefore, $\lim_{x\to 0} f(x) = 0$.

4. (3 points) (a) Show that $\lim_{x\to 3} \frac{2x+3}{4x-9} = 3$.
   (b) Determine if $\lim_{x\to 0^+} \sin(x^{-1})$ exists.
   (c) Determine if $\lim_{x\to 0} x\sin(x^{-2})$ exists.

Solution:

(a) Let $f(x) = \frac{2x+3}{4x-9}$. For any $\epsilon > 0$, we choose $\delta = \min\left\{\frac{\epsilon}{10}, \frac{1}{2}\right\}$. Then for $|x-0| < \delta$, we have that

$$|f(x) - 3| = \left|\frac{10x-30}{4x-9}\right| \leq 10\,|x-3| < \epsilon.$$

Therefore, $\lim_{x\to 0} f(x) = 3$.

(b) Let $f(x) = \sin(x^{-1})$, $a_n = \frac{1}{2n\pi}$ and $b_n = \frac{1}{2n\pi+\frac{\pi}{2}}$. Note that for all $n \in \mathbb{N}$,

$$f(a_n) = \sin(2n\pi) = 0$$

and

$$f(b_n) = \sin(2n\pi + \frac{\pi}{2}) = 1.$$

Then $\lim_{n\to\infty} f(a_n) = 0 \neq 1 = \lim_{n\to\infty} f(b_n)$. Since both a_n and b_n converge to 0 from the right hand side, we have that $\lim_{x\to 0^+} \sin(x^{-1})$ does not exist.

(c) Let $f(x) = x\sin(x^{-2})$. For any $\epsilon > 0$, we choose $\delta = \epsilon$. Then for $|x-0| < \delta$, we have that

$$|f(x) - 0| = \left|x\sin(x^{-2})\right| = |x|\left|\sin(x^{-2})\right| \leq |x| < \epsilon.$$

Therefore, $\lim_{x\to 0} f(x) = 0$.