THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2021)
Suggested Solution of Homework 2

If you find any errors or typos, please email me at
yzwang@math.cuhk.edu.hk

1. (2 points) Let $x_n = \sqrt{n+1} - \sqrt{n}$ for all $n \in \mathbb{N}$. Show that x_n and $\sqrt{n}x_n$ both converges.

Solution: For all $n \in \mathbb{N}$, we have that

$$0 \le x_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n} + \sqrt{n+1}} \le \frac{1}{2\sqrt{n}}.$$

By squeeze theorem, to prove $\lim_{n\to\infty} x_n = 0$, it suffices to show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

For any $\epsilon > 0$, by the Archimedean property, we can find $N \in \mathbb{N}$ such that $N > \epsilon^{-2}$. Then for $n > N$,

$$\left|\frac{1}{\sqrt{n}}\right| < \frac{1}{\sqrt{N}} < \epsilon.$$

Hence x_n converges and $\lim_{n\to\infty} x_n = 0$.

As for $\sqrt{n}x_n$, note that

$$\sqrt{n}x_n = \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n+1}} = (1 + \frac{\sqrt{n+1}}{\sqrt{n}})^{-1}.$$

Since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0,$$

and

$$\frac{\sqrt{n+1}}{\sqrt{n}} - 1 = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} = x_n \cdot \frac{1}{\sqrt{n}},$$

we have that

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1.$$

Therefore,

$$\lim_{n\to\infty} \sqrt{n}x_n = \lim_{n\to\infty} (1 + \frac{\sqrt{n+1}}{\sqrt{n}})^{-1} = \frac{1}{2}.$$

2. (3 points) (a) Suppose x_n is a sequence of positive number such that

$$\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} > 1.$$

Show that x_n is an unbounded sequence.

(b) Suppose x_n is a sequence of positive number such that

$$\lim_{n\to+\infty} x_n^{1/n} < 1.$$

Show that x_n is convergent. If the limit is 1, what can we conclude? Justify your answer.

Solution:

(a) Since $\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} > 1$, then $\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} > r$ for some $r > 1$. It follows that there exists $N \in \mathbb{N}$ such that $\frac{x_{n+1}}{x_n} > r$ for $n \geq N$.

Suppose that x_n is a bounded sequence. Then there exists some $M \in \mathbb{R}$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. Let $t = \log_r \frac{M}{x_N}$. By the Archimedean property, we can find $K \in \mathbb{N}$ such that $K > t$. Then

$$x_{N+K} = x_N \prod_{i=N}^{N+K-1} \frac{x_{i+1}}{x_i} > x_N r^K > x_N r^t > x_N \frac{M}{x_N} = M.$$

Contradiction arises. Therefore x_n must be unbounded.

(b) Since $\lim_{n\to+\infty} x_n^{1/n} < 1$, then $\lim_{n\to+\infty} x_n^{1/n} < s$ for some $s < 1$. It follows that there exists $N_0 \in \mathbb{N}$ such that $x_n^{1/n} < s$ for $n \geq N_0$. Hence $0 < x_n < s^n$ for $n \geq N_0$. Since $s < 1$, we have that $\lim_{n\to\infty} s^n = 0$. By squeeze theorem, we have that $\lim_{n\to\infty} x_n = 0$.

If the limit is 1, we cannot draw any conclusion. Let $a_n = 1$ and $b_n = (1 + \frac{1}{\sqrt{n}})^n$. We have that

$$\lim_{n\to+\infty} a_n^{1/n} = \lim_{n\to+\infty} b_n^{1/n} = 1.$$

But a_n is convergent while $b_n \geq 1 + \frac{n}{\sqrt{n}} = 1 + \sqrt{n} \to \infty$ as $n \to \infty$.

3. (3 points) (a) Let $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for all $n \in \mathbb{N}$. Determine whether $\{x_n\}$ is convergent or not.

   (b) Let $x_1 = 1$ and $x_{n+1} = \sqrt{2x_n}$ for all $n \in \mathbb{N}$, show that $\{x_n\}$ is convergent.

Solution:

(a) We first prove by induction that $x_n \geq 1$ for all $n \in \mathbb{N}$.

When $n = 1$, $x_1 = 1 \geq 1$.

Suppose $x_k \geq 1$ for some $k \in \mathbb{N}$. Then $x_{k+1} = x_k + \frac{1}{x_k} \geq 1$. Hence $x_n \geq 1$ for all $n \in \mathbb{N}$.

Suppose $\{x_n\}$ is convegent. Then $\lim_{n\to\infty} x_n = L \geq 1$. By taking limits on both sides of the equation $x_{n+1} = x_n + \frac{1}{x_n}$, since $L > 0$, we have that $L = L + \frac{1}{L} > L$. Contradiction arises. Therefore $\{x_n\}$ is not convegent.

(b) We prove by induction that $x_n < 2$ and $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

When $n = 1$, $x_1 = 1 < 2$ and $x_2 =\sqrt{2} > x_1$.

Suppose $x_k < 2$ and $x_{k+1} > x_k$ for some $k \in \mathbb{N}$. Then $x_{k+1} = \sqrt{2x_n} < 2$ and $x_{k+2} = \sqrt{2x_{k+1}} > \sqrt{2x_k} = x_{k+1}$. Hence $x_n < 2$ and $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

Therefore $\{x_n\}$ is monotone increasing and bounded from above. It follows that $\{x_n\}$ is convergent.

4. (2 points) If every subsequence of $\{x_n\}$ has a sub-subsequence converging to 0, show that $x_n \to 0$.

Solution: We prove the contrapositive of the statement. Suppose that $\{x_n\}$ does not converge to 0. Then there exists some $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$, we can find $N_n > n$ such that $|x_{N_n}| \geq \epsilon_0$. Note that $\{x_{N_n} : n \in \mathbb{N}\}$ is a subsequence of $\{x_n\}$ satisfying that $|x_{N_n}| \geq \epsilon_0$ for all $n \in \mathbb{N}$. Therefore no subsequence of $\{x_{N_n} : n \in \mathbb{N}\}$ converges to 0.

In conclusion, if every subsequence of $\{x_n\}$ has a sub-subsequence converging to 0, then $x_n \to 0$.