2010 Homework 4 Suggested Solution

December 2020

1. Let the radius and the height be r and h respectively.

   Then they satisfy $r^2 + \left(\frac{h}{2}\right)^2 = a^2$ and the expression to be maximized is $2\pi rh$.

   So, the Lagrange multiplier is $\Phi(r, h, \lambda) = 2\pi rh + \lambda\left(r^2 + \frac{h^2}{4} - a^2\right)$.

   $\Phi_r = 2\pi h + 2\lambda r$

   $\Phi_h = 2\pi r + \frac{\lambda h}{2}$

   $\Phi_\lambda = r^2 + \frac{h^2}{4} - a^2$

   Setting all of them to be zero, we get $r = \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}, h = \sqrt{2}a$

   Area $= 2\pi rh = 2\pi a^2$

2. Lagrange multiplier: $\Phi(x, y, z, \lambda) = xyz + \lambda(x + y + z^2 - 16)$

   $\Phi_x = yz + \lambda$

   $\Phi_y = xz + \lambda$

   $\Phi_z = xy + 2\lambda z$

   $\Phi_\lambda = x + y + z^2 - 16$

   Setting all of them to be zero, we get $x = y = \frac{32}{5}, z = \frac{4}{\sqrt{5}}$

   The product $xyz = \frac{4096}{25\sqrt{5}} = \frac{4096\sqrt{5}}{125} \approx 73.271475$

3. (a) Lagrange multiplier: $\Phi(x, y, z, \lambda_1, \lambda_2) = xyz + \lambda_1(x + y + z - 40) + \lambda_2(x + y - z)$

   $\Phi_x = yz + \lambda_1 + \lambda_2$

   $\Phi_y = xz + \lambda_1 + \lambda_2$

   $\Phi_z = xy + \lambda_1 - \lambda_2$

   $\Phi_{\lambda_1} = x + y + z - 40$

$\Phi_{\lambda_2} = x + y - z$
Setting all of them to be zero, we get $x = y = 10, z = 20$
Hence $w = 2000$

(b) The constraints imply that $x + y = 20$ and $z = 20$.
So the product could be written as $w = 20xy$
Consider in the xy-plane.
Different values of the product w correspond to different hyperbola. The larger the product, the further is the hyperbola is from the origin. Hence, geometrically we could see that to maximize the product w, the corresponding hyperbola must touch the line $x + y = 20$. That is, $x = y = 10$.

(a) $f(x, y) = \ln(2x + y + 1)$
$f(0, 0) = 0$

$$f_x = \frac{2}{2x + y + 1}$$
$f_x(0, 0) = 2$
$$f_y = \frac{1}{2x + y + 1}$$
$f_y(0, 0) = 1$

$$f_{xx} = -\frac{4}{(2x + y + 1)^2}$$
$f_{xx}(0, 0) = -4$
$$f_{xy} = -\frac{2}{(2x + y + 1)^2}$$
$f_{xy}(0, 0) = -2$
$$f_{yy} = -\frac{1}{(2x + y + 1)^2}$$
$f_{yy}(0, 0) = -1$

$$f_{xxx} = \frac{16}{(2x + y + 1)^3}$$
$f_{xxx}(0, 0) = 16$
$$f_{xxy} = \frac{8}{(2x + y + 1)^3}$$
$f_{xxy}(0, 0) = 8$
$$f_{xyy} = \frac{4}{(2x + y + 1)^3}$$
$f_{xyy}(0, 0) = 4$

$f_{yyy} = \dfrac{2}{(2x + y + 1)^3}$

$f_{yyy}(0, 0) = 2$

Quadratic approximation:

$f(x, y) \approx 2x + y - 2x^2 - 2xy - \dfrac{1}{2}y^2$

Cubic approximation:

$f(x, y) \approx \text{(quad approx)} + \dfrac{8}{3}x^3 + 4x^2y + 2xy^2 + \dfrac{1}{3}y^3$

(b) Differentiate as in part (a).

The answer is:

Quadratic approximation:

$f(x, y) \approx 1 + x + y + x^2 + xy + y^2$

Cubic approximation:

$f(x, y) \approx \text{(quad approx)} + x^3 + x^2y + xy^2 + y^3$