# Solution to Midterm 1

1. (a) Consider the characteristic polynomial of $A$.

$$\det(A - tI) = \left(\frac{\sqrt{3}}{2} - t\right)^2 + \frac{1}{4}$$

Obviously, the above polynomial does not split over $\mathbb{R}$, we see that $A$ is not diagonalizable over $\mathbb{R}$.

(b) However, the characteristic polynomial of $A$ does split over $\mathbb{C}$. Moreover, we have two distinct eigenvalues of $A$. Hence, $A$ is diagonalizable over $\mathbb{C}$.

(c) From the above, see that the eigenvalues of $A$ are $\frac{\sqrt{3}-i}{2}$ and $\frac{\sqrt{3}+i}{2}$. As $A$ is diagonalizable over $\mathbb{C}$, there is some invertible matrix $Q \in M_{2\times 2}(\mathbb{C})$ such that

$$A = Q \begin{pmatrix} \frac{\sqrt{3}-i}{2} & 0 \\ 0 & \frac{\sqrt{3}+i}{2} \end{pmatrix} Q^{-1}.$$

So we have the following.

$$A^k = Q \begin{pmatrix} \left(\frac{\sqrt{3}-i}{2}\right)^k & 0 \\ 0 & \left(\frac{\sqrt{3}+i}{2}\right)^k \end{pmatrix} Q^{-1}$$

Using the fact that $\left(\frac{\sqrt{3}\pm i}{2}\right)^3 = \pm i$, we see that the smallest positive integer $k \in \mathbb{N}$ is 12.

$$A^{12} = Q \begin{pmatrix} \left(\frac{\sqrt{3}-i}{2}\right)^{12} & 0 \\ 0 & \left(\frac{\sqrt{3}+i}{2}\right)^{12} \end{pmatrix} Q^{-1} = QIQ^{-1} = I$$

2. (a) Note that

$$P = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

So we may choose $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ as a basis.

(b) Observe that $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is the normal vector to the plane $2x-y+2z=0$.

Denote

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

So we may choose $\gamma = \{v_1, v_2, v_3\}$ as a basis for $\mathbb{R}^3$. As $T$ is the reflection about the plane $P$, one can easily check that $\gamma$ is actually a basis consisting of eigenvectors of $T$.

$$T(v_1) = v_1, \quad T(v_2) = v_2, \quad T(v_3) = -v_3$$

Hence, $T$ is diagonalizable.

(c) Using

$$[T]_\beta = [I]_\gamma^\beta [T]_\gamma [I]_\beta^\gamma,$$

with $[I]_\gamma^\beta = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ and $[I]_\beta^\gamma = \left([I]_\gamma^\beta\right)^{-1}$, we have

$$[T]_\beta = \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix}.$$

3. Let

$$w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\beta = \{w_1, w_2, w_3\}$. Note that $\beta$ is a basis for $W$ and we have

$$T(w_1) = w_1 + 2w_3, \quad T(w_2) = 2w_1 + w_2, \quad T(w_3) = 2w_2 + w_3.$$

So we see that

$$[T]_\beta = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

(a) Let $f(t)$ be the characteristic polynomial of $T$.

$$f(t) = \det([T]_\beta - tI) = -t^3 + 3t^2 - 3t + 9$$

(b) By Cayley-Hamilton Theorem, we have $f(T) = 0$.

$$f(T) = -T^3 + 3T^2 - 3T + 9I = 0$$

After rearranging the above, we have

$$I = \frac{1}{9}T(T^2 - 3T + 3I) = \frac{1}{9}(T^2 - 3T + 3I)T.$$

Hence, we see that $T$ is invertible and $T^{-1} = T^2 - 3T + 3I$.

4. Let $W$ be the $T$-cyclic subspace of $V$ generated by $v$. As $T^2v = -v$, we see that $W = \text{span}\{v, Tv\}$. Note that $v$ and $Tv$ are linearly independent (otherwise we have $Tv = cv$ and $c^2 = -1$, which is not possible over $\mathbb{R}$). So $\beta = \{v, Tv\}$ is a basis for $W$. Consider $T_W$, we have

$$[T_W]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the characteristic polynomial of $T_W$, say $f_W(t)$, is

$$f_W(t) = \det([T_W]_\beta - tI) = t^2 + 1.$$

As $f_W(t)$ does not split over $\mathbb{R}$, the characteristic polynomial of $T$, say $f(t)$, does not split over $\mathbb{R}$ too (since $f_W(t)$ is a factor of $f(t)$), which means $T$ is not diagonalizable over $\mathbb{R}$.

5. Suppose $A$ has $n$ distinct positive real eigenvalues, say $\lambda_1, \lambda_2, \ldots, \lambda_n$. Note that the characteristic polynomial of $A$ splits and $\lambda_i$ are distinct.

$$f(t) = (\lambda_1 - t)(\lambda_2 - t) \cdots (\lambda_n - t)$$

Hence, $A$ is diagonalizable. Then there exists invertible matrix $Q$ and diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

such that $A = QDQ^{-1}$. As $\lambda_i$ are positive, we can choose

$$C = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix}$$

and $C^2 = D$. So we can let $B = QCQ^{-1}$ and we have

$$B^2 = QCQ^{-1}QCQ^{-1} = QC^2Q^{-1} = QDQ^{-1}.$$

6. Obviously, we have $W \subset V$. We want to show $V \subset W$. For any $v \in V$, there exists $v_1 \in V$ and $w_1 \in W$ such that

$$v = w_1 + T(v_1) + T^2(v_1).$$

Again for this $v_1 \in V$, there exists $v_2 \in V$ and $w_2 \in W$ such that $v_1 = w_2 + T(v_2) + T^2(v_2)$. So we have

$$\begin{aligned} v &= w_1 + T(w_2 + T(v_2) + T^2(v_2)) + T^2(w_2 + T(v_2) + T^2(v_2)) \\ &= w_1 + (T + T^2)(w_2) + (T^2 + 2T^3 + T^4)(v_2) \end{aligned}$$

Repeat this process $n$ times, we will get

$$v = w_1 + f(T)(w_n) + g(T)(v_n),$$

where $f(T)$ and $g(T)$ are polynomials of $T$. In particular, $g(T)$ consists of terms $a_j T^j$ with $n \leq j \leq 2n$, so $g(T)$ is the zero transformation on $V$. As $W$ is $T$-invariant, $f(T)(w_n) \in W$, so $v = w_1 + f(T)(w_n) + g(T)(v_n) \in W$.