Solution to Homework 11

Sec. 7.1

2. (c) First, we consider the characteristic polynomial of $A = \begin{pmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{pmatrix}$.

$$\begin{aligned}
\det(A-\lambda I) &= \det\begin{pmatrix} 11-\lambda & -4 & -5 \\ 21 & -8-\lambda & -11 \\ 3 & -1 & -\lambda \end{pmatrix} \\
&= \det\begin{pmatrix} 11-\lambda-12 & 0 & -5+5\lambda \\ 21-3(8+\lambda) & 0 & -11+\lambda(8+\lambda) \\ 3 & -1 & -\lambda \end{pmatrix} \\
&= -(x-2)^2(x+1)
\end{aligned}$$

Hence, we see that $\lambda_1 = -1$ and $\lambda_2 = 2$ are two eigenvalues of A with multiplicities 1 and 2 respectively. Consider $\lambda_1 = -1$, we have

$$K_{\lambda_1} = E_{\lambda_1} = N(A+I) = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

For $\lambda_2 = 2$, we have

$$K_{\lambda_2} = N((A-2I)^2) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Then we can pick, for example $v = (0,1,-1)^t$ such that $(A-2I)v \neq 0$ but $(A-2I)^2v = 0$. Since $(A-2I)v = (1,1,1)^t$, we obtain the cycle of generalized eigenvectors. Then we obtain a Jordan canonical basis β for A.

$$\beta = \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

So we have the following Jordan canonical form J of A.

$$J = [L_A]_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

3. (a) Let $\gamma = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$. Then we see that the matrix representation of T under γ is

$$[T]_\gamma = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Again, we look at the characteristic polynomial of T. Since $[T]_\gamma$ is upper-triangular, we have

$$\det\left([T]_\gamma - \lambda I\right) = (2 - \lambda)^3.$$

So, we see that $\lambda = 2$ is an eigenvalue. However, we have $\dim(E_\lambda) = 1$, which means the desired basis is a single cycle of length 3. Check the following nullspaces.

$$N\left([T]_\gamma - 2I\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$N\left(([T]_\gamma - 2I)^2\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

$$N\left(([T]_\gamma - 2I)^3\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Then we can choose $v = (0, 0, 1)^t$ such that $([T]_\gamma - 2I)v \neq 0$ and $([T]_\gamma - 2I)^2 v \neq 0$, but $([T]_\gamma - 2I)^3 v = 0$. Hence, the basis could be $\beta = \left\{(T - 2I_o)^2(x^2), (T - 2I_o)(x^2), x^2\right\}$ where I_o denotes the identity operator on $P_2(\mathbb{R})$. In other words, we have

$$\beta = \left\{2, -2x, x^2\right\}$$

and the Jordan canonical form J of T is

$$J = [T]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. Let $W = \text{span}(\gamma)$. Note that W is $(T - \lambda I)$-invariant by the definition of a cycle. Hence, for any $w \in W$, we have the following.

$$T(w) = (T - \lambda I)(w) + \lambda I(w) = (T - \lambda I)(w) + \lambda w \in W$$

Thus, W is also T-invariant.

5. Suppose the initial eigenvectors are distinct. If the cycles are not disjoint, then we have some element x in at least two cycles, say γ_1 and γ_2. Consider the smallest integer q such that

$$(T - \lambda I)^q(x) = 0.$$

We see that $(T-\lambda I)^{q-1}(x)$ is the initial eigenvector for both γ_1 and γ_2, which is a contradiction. Hence, the cycles must be distinct.

6. (a) Obviously, we have $T(x) = 0$ if and only if $(-T)(x) = 0$, which means $N(T) = N(-T)$.

   (b) Using the fact that $(-T)^k = (-1)^k T^k$ and the result from (a), we have the following.

$$N((-T)^k) = N((-1)^k T^k) = N(T^k)$$

   (c) Using the fact that $(\lambda I - T) = -(T - \lambda I)$ and the result from (b), we have the following.

$$N\left((\lambda I - T)^k\right) = N\left(((-(T-\lambda I))^k\right) = N\left((T-\lambda I)^k\right)$$

13. For each i, let J_i be the Jordan canonical form of the restriction of T to K_{λ_i}. So, we can find some basis β_i for K_{λ_i} such that

$$\left[T\big|_{K_{\lambda_i}}\right]_{\beta_i} = J_i.$$

We see that $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ will be a basis for V as V is a direct sum of K_{λ_i}. Then, one can easily check that $[T]_\beta = J_1 \oplus J_2 \oplus \cdots \oplus J_k$ is the Jordan canonical form of T.

Sec. 7.2

4. (c) First, we consider the characteristic polynomial of $A = \begin{pmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{pmatrix}$.

$$\begin{aligned}
\det(A-\lambda I) &= \det\begin{pmatrix} -\lambda & -1 & -1 \\ -3 & -1-\lambda & -2 \\ 7 & 5 & 6-\lambda \end{pmatrix} \\
&= \det\begin{pmatrix} -\lambda & -1 & -1 \\ -3+2\lambda & 1-\lambda & 0 \\ 7-\lambda(6-\lambda) & \lambda-1 & 0 \end{pmatrix} \\
&= -(\lambda-1)(\lambda-2)^2
\end{aligned}$$

For $\lambda_1 = 2$, we see that the multiplicity is 2, so we check the null space of $A - 2I$ and $(A-2I)^2$.

$$N(A-2I) = N\begin{pmatrix} -2 & -1 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}\right\}$$

$$N((A-2I)^2) = N\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & -2 & -1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}\right\}$$

Obviously, we can choose $v = (2, -1, 0)$ as $(A - 2I)v \neq 0$ and $(A - 2I)^2 v = 0$. Then, we can get

$$\beta_1 = \{(A - 2I)v, v\} = \left\{ \begin{pmatrix} -3 \\ -3 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

For $\lambda_2 = 1$, we just check the null space of $A - I$.

$$N(A - I) = N \begin{pmatrix} -1 & -1 & -1 \\ -1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Then, we can choose

$$\beta_2 = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Let $\beta = \beta_1 \cup \beta_2$ and

$$Q = \begin{pmatrix} -3 & 2 & 0 \\ -3 & -1 & 1 \\ 9 & 0 & -1 \end{pmatrix}.$$

Finally, we have the Jordan canonical form of A.

$$J = Q^{-1}AQ = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Again, we first consider the characteristic polynomial of A.

$$\det(A - \lambda I) = \lambda^4 - 4\lambda^3 + 4\lambda^2 = \lambda^2(\lambda - 2)^2$$

For $\lambda_1 = 0$, we check the null space of A.

$$N(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Since we only have one eigenvector, but $\lambda_1 = 0$ is of multiplicity 2, so there should one cycle of generalized eigenvectors. Consider the null space of A^2.

$$N(A^2) = \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

Then, we choose some v such that $A^2 = 0$ but $Av \neq 0$. One possible choice is $v = (2, 1, 0, 2)^t$. So, we can choose a basis for K_{λ_1}.

$$\beta_1 = \{Av, v\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

For $\lambda_2 = 2$, we check the null space of $A - 2I$.

$$N(A - 2I) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Since $\dim(E_{\lambda_2}) = 2$, we have $K_{\lambda_2} = E_{\lambda_2}$.

$$\beta_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Let $\beta = \beta_1 \cup \beta_2$ and

$$Q = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}.$$

Finally, we have the Jordan canonical form of A.

$$J = Q^{-1}AQ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

5. (d) Consider the standard basis $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

$$\begin{aligned}
T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= 3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Hence, we have the following matrix representation of T under γ.

$$[T]_\gamma = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Then we consider the characteristic polynomial of T.

$$\det([T]_\gamma - \lambda I) = (\lambda - 2)^3(\lambda - 4)$$

For $\lambda_1 = 2$, we check the null space of $[T]_\gamma - 2I$.

$$N([T]_\gamma - 2I) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

However, the multiplicity of λ_1 is 3, so we have a cycle of generalized eigenvectors of length 3.

$$N(([T]_\gamma - 2I)^2) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$N(([T]_\gamma - 2I)^3) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

Obviously, we can choose $v = (0, 0, 1, 2)^t$ such that $([T]_\gamma - 2I)^3 v = 0$ while $([T]_\gamma - 2I)^2 v \neq 0$ and $([T]_\gamma - 2I)v \neq 0$. So, we have

$$\beta_1 = \left\{ ([T]_\gamma - 2I)^2 v, ([T]_\gamma - 2I)v, v \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

For $\lambda_2 = 4$, we just check the null space of $[T]_\gamma - 4I$.

$$N([T]_\gamma - 4I) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \end{pmatrix} \right\}$$

So, we can take

$$\beta_2 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

Let $\beta = \beta_1 \cup \beta_2$ and

$$Q = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

Finally, we have

$$J = [T]_\beta = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

6. Note that for any eigenvalue λ of A, we have

$$(A^t - \lambda I)^r = \left((A - \lambda I)^t\right)^r = ((A - \lambda I)^r)^t$$

for any positive integer r, which means that

$$\operatorname{rank}((A - \lambda I)^r) = \operatorname{rank}((A^t - \lambda I)^r).$$

As a consequence, A and A^t will give the same dot diagram and, hence, give the same Jordan canonical form J. Hence, A and A^t are similar as they are both similar to J.