Solution to Homework 10

Sec. 6.5

2. (c) Consider the characteristic polynomial of $A = \begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$.

$$\det(A - \lambda I) = (2-\lambda)(5-\lambda) - (3-3i)(3+3i) = \lambda^2 - 7\lambda - 8$$

By solving $\det(A - \lambda I) = 0$, we have $\lambda = -1$ or $\lambda = 8$.
For $\lambda = -1$, we have

$$N(A + I) = \text{span}(\{(-1+i, 1)^t\}).$$

For $\lambda = 8$, we have

$$N(A - 8I) = \text{span}(\{(1, 1+i)^t\}).$$

By normalizing the two directions, we have

$$P = \frac{1}{\sqrt{3}} \begin{pmatrix} -1+i & 1 \\ 1 & 1+i \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 8 & 0 \end{pmatrix}.$$

4. Note that $[T]_\beta = \begin{pmatrix} z \end{pmatrix}$, where β is the standard basis for $\mathbb{C}^1$ (orthonormal). Then we have $[T*]_\beta = \begin{pmatrix} \overline{z} \end{pmatrix}$ and $T_z^* = T_{\overline{z}}$. In other words, $T_z^*(u) = \overline{z}u$. Hence, we see that T_z is always normal, self-adjoint when z is real, and unitary when $|z| = 1$.

5. (c) Consider the characteristic polynomial of the matrix on the left, one can easily check that 1 and $\pm i$ are the eigenvalues. While for the matrix on the right, the eigenvalues are, obviously, -1, 0 and 2. Hence, They are not unitarily equivalent.

7. Suppose T is unitary. There exists an orthonormal basis β such that $T(\beta)$ is an orthonormal basis for V. In other words, we have

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $|\lambda_i| = 1$. Then, by defining μ_i such that $\mu_i^2 = \lambda_i$, we have $|\mu_i| = 1$. Let

$$D = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mu_n \end{pmatrix}$$

and define U such that $[U]_\beta = D$. Then we see that $U^2 = T$ and U is a unitary operator.

9. Consider $V = \mathbb{R}^2$ and $U : V \to V$ with $U(a, b) = (a + b, 0)$. Let $\beta = \{(1, 0), (0, 1)\}$ be an orthonormal basis for V. Then we see that

$$\|U(1, 0)\| = \|U(0, 1)\| = \|(1, 0)\| = \|(0, 1)\| = 1.$$

However, $\|U(1, 1)\| = 2 \neq \sqrt{2} = \|(1, 1)\|$. Hence, we see that U may not be unitary.

13. Consider $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So, A and B are similar.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

But they are not unitarily equivalent as A is symmetric while B is not.

15. (a) Since U is unitary, we have $\|U_W(x)\| = \|U(x)\| = \|x\|$. This means that U_W is injective. As W is of finite dimension, U_W is also surjective by considering the rank and nullity. Hence, $U(W) = W$.

(b) Note that $V = W \oplus W^\perp$ and $U(x) \in V$. For any $x \in W^\perp$,

$$U(x) = w + y$$

for some $w \in W$ and $y \in W^\perp$. To show that $W^\perp$ is U-invariant, we need to show $w = 0$. From (a), we see that U_W is surjective, so there is some $v \in W$ such that $U(v) = w$. As U is unitary, we have $\|v\| = \|w\|$. Similarly, we have

$$\|x\|^2 = \|U(x)\|^2 = \|w + y\|^2 = \|w\|^2 + \|y\|^2,$$

where the last equality is by the orthogonality of w and y. Besides, we get $U(x + v) = 2w + y$ and

$$\|x\|^2 + \|v\|^2 = \|x + v\|^2 = \|2w + y\|^2 = 4\|w\|^2 + \|y\|^2.$$

Then one can easily solve that $\|w\|^2 = 0$, which means $w = 0$.

16. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for V. Consider an operator U defined by

$$\begin{cases} U(e_1) = e_2 & \\ U(e_{2i+1}) = e_{2i-1} & \text{for } i \geq 1 \\ U(e_{2i}) = e_{2i+2} & \text{for } i \geq 1 \end{cases}$$

One can easily check that U is unitary.

$$\|U(x)\| = \sum_{i=1}^{\infty} \alpha_i = \|x\| \text{, where } x = \sum_{i=1}^{\infty} \alpha_i e_i$$

Then for the subspace $W = \mathrm{span}(\{e_2, e_4, e_6, \dots\})$, we see that W is U-invariant. However, $W^\perp$ is not U-invariant as $U(e_1) = e_2 \notin W^\perp$.

Sec. 6.6

4. Suppose T is the orthogonal projection of V on W. Then we have

$$R(T)^\perp = N(T), \quad N(T)^\perp = R(T)$$

and $R(T) = W$. If we have $R(I-T) = N(T)$ and $N(I-T) = R(T)$, then $R(I-T)^\perp = N(I-T)$ and $N(I-T)^\perp = R(I-T)$, which means $I-T$ is an orthogonal projection. With $T = T^2$, we have the following.

For any $(I-T)(x) \in R(I-T)$, we have

$$T(I-T)(x) = T(x) - T^2(x) = T(x) - T(x) = 0,$$

so $(I-T)(x) \in N(T)$. If $x \in N(T)$, then we have

$$x = (I-T)(x) \in R(I-T).$$

Hence, $R(I-T) = N(T)$.

For any $x \in N(I-T)$, we have $(I-T)(x) = 0$, which means

$$x = T(x) \in R(T).$$

If $T(x) \in R(T)$, then we have

$$(I-T)(T(x)) = T(x) - T^2(x) = T(x) - T(x) = 0,$$

so $T(x) \in N(I-T)$. Hence, $N(I-T) = R(T)$.

With the above, we have the following.

$$R(I-T)^\perp = N(T)^\perp = R(T) = N(I-T)$$

$$N(I-T)^\perp = R(T)^\perp = N(T) = R(I-T)$$

In other words, $I-T$ is an orthogonal projection. Moreover, we have $R(I-T) = N(T) = R(T)^\perp = W^\perp$, so $I-T$ is the orthogonal projection of V on $W^\perp$.

6. Let T be a projection of a finite-dimensional inner product space. We need to show that $R(T)^\perp = N(T)$ and $N(T)^\perp = R(T)$. For any $x \in R(T)^\perp$, we have

$$\langle T(x), T(x)\rangle = \langle x, T^*(T(x))\rangle = \langle x, T(T^*(x))\rangle = 0,$$

which means $T(x) = 0$ and $x \in N(T)$. If $x \in N(T)$, then $T(x) = 0$, which means x is an eigenvector of T with respect to eigenvalue 0. Since T is normal, x is an eigenvector of T^* with respect to eigenvalue 0, too. Then for any $T(y) \in R(T)$, we have

$$\langle x, T(y)\rangle = \langle T^*(x), y\rangle = 0.$$

Hence, $R(T)^\perp = N(T)$. Since the space is of finite dimension, we have

$$N(T)^\perp = \left(R(T)^\perp\right)^\perp = R(T).$$

Thus, T is an orthogonal projection.

7. (a) Using the fact that $T_iT_j = \delta_{ij}T_j$, we have

$$\begin{aligned} g(T) &= g\left(\sum_{i=1}^k \lambda_i T_i\right) \\ &= \sum_j \alpha_j \left(\sum_{i=1}^k \lambda_i T_i\right)^j \\ &= \sum_j \alpha_j \left(\sum_{i=1}^k \lambda_i^j T_i\right) \\ &= \sum_{i=1}^k \left(\sum_j \alpha_j \lambda_i^j\right) T_i = \sum_{i=1}^k g(\lambda_i) T_i. \end{aligned}$$

(b) Similarly, by $T_iT_j = \delta_{ij}T_j$, we have

$$T_0 = T^n = \sum_{i=1}^k \lambda_i^n T_i.$$

For any eigenvector v_i with respect to eigenvalue λ_i, we have

$$0 = T_0(v_i) = T^n(v_i) = \left(\sum_{i=1}^k \lambda_i^n T_i\right)(v_i) = \lambda_i^n v_i,$$

which means $\lambda_i = 0$. Since this is true for all i, we have

$$T = \sum_{i=1}^k \lambda_i T_i = T_0.$$

(c) Suppose U commutes with each T_i. Then we have

$$\begin{aligned} UT &= U\left(\sum_{i=1}^{k} \lambda_i T_i\right) \\ &= \sum_{i=1}^{k} \lambda_i U T_i \\ &= \sum_{i=1}^{k} \lambda_i T_i U \\ &= \left(\sum_{i=1}^{k} \lambda_i T_i\right) U = TU \end{aligned}$$

Conversely, suppose U commutes with T. Note that for each T_i, there exists some polynomial g_i such that $g_i(T) = T_i$. Then we have

$$UT_i = Ug_i(T) = g_i(T)U = T_iU.$$

(d) Note that $T_iT_j = \delta_{ij}T_j$ and $T = \sum_{i=1}^{k} \lambda_i T_i$. Let

$$U = \sum_{i=1}^{k} \lambda_i^{\frac{1}{2}} T_i.$$

Then one can easily check that $U^2 = T$. Since T_i are self-adjoint, that is T_i is normal, thus U is normal, too.

(e) Note that V is finite-dimensional. So T is invertible if and only if $N(T) = 0$. But this means 0 is not an eigenvalue of T.

(f) Suppose T is a projection of V on W along $W^{\perp}$. Let λ be eigenvalue and $v \in V$ be the corresponding eigenvector. Then there is some $w \in W$ and $y \in W^{\perp}$ such that $v = w + y$. So, we have

$$\begin{aligned} w = T(w+y) &= \lambda(w+y) \\ (1-\lambda)w &= \lambda y. \end{aligned}$$

This means that λ can only be 1 or 0.

(g) Suppose $T = -T^*$. Note that if λ_i is an eigenvalue of T, then $\overline{\lambda_i}$ will be an eigenvalue of T^*. Let v_i be the eigenvector with respect to eigenvalue λ_i. It follows that

$$\lambda_i v_i = Tv_i = -T^*v_i = -\overline{\lambda_i}v_i,$$

which means every λ_i is an imaginary number. Conversely, if every λ_i is an imaginary number, then $\overline{\lambda_i} = -\lambda_i$. Note that T_i is self-adjoint. Then we have

$$T^* = \left(\sum_{i=1}^{k} \lambda_i T_i\right)^* = \sum_{i=1}^{k} \overline{\lambda_i} T_i^* = \sum_{i=1}^{k} (-\lambda_i) T_i = -T,$$

which means $T = -T^*$.

10. We prove the statement by induction on the dimension of V, $n = \dim(V)$.

When $n = 1$, the statement is trivial. Now suppose the statement holds for $n \leq k - 1$, we consider $n = k$.

Pick an arbitrary eigenspace $W = E_\lambda$ of T with respect to some eigenvalue λ. Obviously, W is T-invariant. Note that W is also U-invariant as $U(w)$ is an eigenvector of T with respect to eigenvalue λ.

$$TU(w) = UT(w) = \lambda U(w)$$

If $W = V$, by Theorem 6.17, as U is self-adjoint, we may find an orthonormal basis β for V consisting of eigenvectors of U. But β are also orthonormal eigenvectors of T, so the result follows.

On the other hand, if W is a proper subspace of V, we may find β in following way.

Note that T_W and U_W are normal by Exercise 7 and 8 of Section 6.4. Using the induction hypothesis, we may choose an orthonormal basis β_1 for W consisting of eigenvectors of T_W and U_W, which are eigenvectors of T and U too.

Similarly, we see that $W^\perp$ is T^*-invariant and U^*-invariant. But T and U are normal, so $W^\perp$ is T-invariant and U-invariant. Again, by the induction hypothesis, we may choose an orthonormal basis β_2 for $W^\perp$ consisting of eigenvectors of T and U.

Let $\beta = \beta_1 \cup \beta_2$. As V is of finite dimension, we see that β is a basis for V consisting of eigenvectors of T and U. Hence, the statement is also true for $n = k$.