Solution to Homework 9

Sec. 6.3

12. (a) For any $x \in R(T^*)^\perp$, we have $\langle x, T^*(y)\rangle = 0$ for all $y \in V$. Then we have
$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle = 0$$
for all $y \in V$. In other words, we have $T(x) = 0$ and $x \in N(T)$. Conversely, for any $x \in N(T)$, we have
$$\langle x, T^*(y)\rangle = \langle T(x), y\rangle = \langle 0, y\rangle = 0$$
for any $y \in V$. So we have $x \in R(T^*)^\perp$.

    (b) By Exercise 13 (c) of Section 6.2, one can show that $W = \left(W^\perp\right)^\perp$ for any finite-dimensional subspace W. Then we have
$$R(T^*) = \left(R(T^*)^\perp\right)^\perp = N(T)^\perp.$$

14. First, we show that T is linear. For any $x_1, x_2 \in V$ and $c \in \mathbb{F}$, we have
$$\begin{aligned} T(x_1 + cx_2) &= \langle x_1 + cx_2, y\rangle z \\ &= \langle x_1, y\rangle z + c\langle x_2, y\rangle z \\ &= T(x_1) + cT(x_2) \end{aligned}$$
So T is a linear operator on V and T^* exists. For any $v \in V$, we have
$$\begin{aligned} \langle x, T^*(v)\rangle &= \langle T(x), v\rangle \\ &= \langle \langle x, y\rangle z, v\rangle \\ &= \langle x, y\rangle \langle z, v\rangle \\ &= \left\langle x, \overline{\langle z, v\rangle} y\right\rangle \\ &= \langle x, \langle v, z\rangle y\rangle \end{aligned}$$
Since this is true for any $x \in V$, we have $T^*(v) = \langle v, z\rangle y$.

Sec. 6.4

2. (c) Let β be the standard basis. Then we have the following.

$$[T]_\beta = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix}$$

It is easy to check that T is normal but not self-adjoint. So we can obtain an orthonormal basis of eigenvectors of T for V.

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{2} + \frac{1}{2}i \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} - \frac{1}{2}i \end{pmatrix} \right\}$$

(d) By orthogonalizing the standard basis, we obtain an orthonormal basis for $P_2(\mathbb{R})$.

$$\beta = \left\{ 1, \sqrt{3}(2t-1), \sqrt{6}(6t^2 - 6t + 1) \right\}$$

Then we have the following.

$$[T]_\beta = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 6\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

So we see that T is neither normal nor self-adjoint.

4. Suppose T and U are self-adjoint operators. Note that

$$(TU)^* = U^*T^* = UT.$$

So it is easy to see that TU is self-adjoint if and only if $TU = UT$.

7. (a) Note that W is a subspace of V. Suppose T is a self-adjoint linear operator on V. For any $x, y \in W$, we have the following.

$$\begin{aligned} \langle x, (T_W)^*(y) \rangle &= \langle T_W(x), y \rangle \\ &= \langle T(x), y \rangle \\ &= \langle T^*(x), y \rangle \\ &= \langle x, T(y) \rangle \\ &= \langle x, T_W(y) \rangle \end{aligned}$$

So T_W is self-adjoint.

(b) We want to show that for any $y \in W^\perp$, we have $T^*(y) \in W^\perp$. To show that, note that W is T-invariant. Consider any $x \in W$, we have $T(x) \in W$. Then for any $y \in W^\perp$, we have the following.

$$\langle x, T^*(y) \rangle = \langle T(x), y \rangle = 0$$

In other words, $T^*(y) \in W^\perp$ and $W^\perp$ is T^*-invariant.

(c) We want to show that $(T_W)^*(y) = (T^*)_W(y)$ for any $y \in W$.

$$\begin{aligned}\langle x, (T_W)^*(y)\rangle &= \langle T_W(x), y\rangle \\ &= \langle T(x), y\rangle \\ &= \langle x, T^*(y)\rangle \\ &= \langle x, (T^*)_W(y)\rangle\end{aligned}$$

Since this is true for any $x \in W$, the result follows.

(d) From the above part, we see that $(T_W)^* = (T^*)_W$. As T is normal, we have $TT^* = T^*T$. Then we have the following.

$$\begin{aligned}T_W(T_W)^* &= T_W(T^*)_W \\ &= (TT^*)_W \\ &= (T^*T)_W \\ &= (T^*)_W T_W \\ &= (T_W)^* T_W\end{aligned}$$

Hence, we see that T_W is normal.

8. Since T is normal, we see that T is diagonalizable. Suppose W is T-invariant, then, by Exercise 24 of Section 5.4, T_W is also diagonalizable. Then consider a basis for W consisting of eigenvectors of T.

   But these eigenvectors of T are also eigenvectors of T^* as T is normal. In other words, we have a basis for W consisting of eigenvectors of T^*, which means W is also T^*-invariant.

9. By Theorem 6.15 (a), we have $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$, which means that $T(x) = 0$ if and only if $T^*(x) = 0$. Hence, we see that

$$N(T) = N(T^*).$$

   By Exercise 12 of Section 6.2, we have $R(T^*) = N(T)^\perp$. Hence, the result follows.

$$R(T^*) = N(T)^\perp = N(T^*)^\perp = R(T)$$

10. Suppose T is self-adjoint, so $T^* = T$.

$$\begin{aligned}\|T(x) \pm ix\|^2 &= \langle T(x) \pm ix, T(x) \pm ix\rangle \\ &= \|T(x)\|^2 \pm \langle T(x), ix\rangle \pm \langle ix, T(x)\rangle + \|x\|^2 \\ &= \|T(x)\|^2 \pm \bar{i}\,\langle T(x), x\rangle \pm i\,\langle T^*(x), x\rangle + \|x\|^2 \\ &= \|T(x)\|^2 \mp i\,\langle T(x), x\rangle \pm i\,\langle T(x), x\rangle + \|x\|^2 \\ &= \|T(x)\|^2 + \|x\|^2\end{aligned}$$

   From the equality, we see that $\|T(x) - x\| = 0$ if and only if $T(x) = 0$ and $x = 0$. So $T - iI$ is injective. $T - iI$ is also surjective as V is of finite dimension. Hence, $T - iI$ is invertible. Similarly, we also have $T + iI$ to be invertible.

To check that $\left[(T - iI)^{-1}\right]^* = (T + iI)^{-1}$, we have the following.

$$\begin{aligned}
\left\langle x, \left[(T - iI)^{-1}\right]^* (T + iI)(y) \right\rangle &= \left\langle (T - iI)^{-1}(x), (T + iI)(y) \right\rangle \\
&= \left\langle (T - iI)^{-1}(x), (T^* + iI)(y) \right\rangle \\
&= \left\langle (T - iI)^{-1}(x), (T - iI)^*(y) \right\rangle \\
&= \left\langle (T - iI)(T - iI)^{-1}(x), y \right\rangle \\
&= \langle x, y \rangle
\end{aligned}$$

As x is arbitrary, we see that $\left[(T - iI)^{-1}\right]^* (T + iI) = I$. Hence, the result follows.

12. Since the characteristic polynomial of T splits, by Schur's Theorem, there exists an orthonormal basis $\beta = \{v_1, v_2, \ldots, v_n\}$ such that $[T]_\beta$ is upper triangular.

    We want to show that β is an orthonormal basis consisting of eigenvectors of T. Let $[T]_\beta = \left(A_{i,j}\right)$, where $\left(A_{i,j}\right)$ is upper triangular.

    Note that $T(v_1) = A_{1,1}v_1$, so v_1 is an eigenvector of T. Suppose t is the largest integer such that $v_1, v_2, \ldots, v_t$ are all eigenvectors with respect to eigenvalues λ_i.

    If $t = n$, then our claim is done. Suppose not, we see that

    $$T(v_{t+1}) = \sum_{i=1}^{t+1} A_{i,t+1} v_i.$$

    Note that v_i are eigenvectors of T^* with respect to eigenvalues $\overline{\lambda_i}$ and v_i are orthogonal to each other. For $i = 1, 2, \ldots, t$, we have the following.

    $$A_{i,t+1} = \langle T(v_{t+1}), v_i \rangle = \langle v_{t+1}, T^*(v_i) \rangle = \left\langle v_{t+1}, \overline{\lambda_i} v_i \right\rangle = 0$$

    So we have v_{t+1} to be an eigenvector of T too.

    But this is a contradiction, so we must have $t = n$. In other words, β is a basis for V consisting of eigenvectors of T By Theorem 6.17, T is self-adjoint.

14. Suppose U and T are self-adjoint operators on V such that $UT = TU$. We prove the statement by induction on the dimension n of V.

    When $n = 1$, the statement is trivial. Now suppose the statement holds for $n \leq k - 1$, we consider $n = k$.

    Pick an arbitrary eigenspace $W = E_\lambda$ of T with respect to some eigenvalue λ. Obviously, W is T-invariant. Note that W is also U-invariant as $U(w)$ is an eigenvector of T with respect to eigenvalue λ.

    $$TU(w) = UT(w) = \lambda U(w)$$

If $W = V$, by Theorem 6.17, as U is self-adjoint, we may find an orthonormal basis β for V consisting of eigenvectors of U. But β are also eigenvectors of T, so the result follows.

On the other hand, if W is a proper subspace of V, we may find β in following way.

Note that T_W and U_W are self-adjoint by Exercise 7 (a) of Section 6.4. Using the induction hypothesis, we may choose an orthonormal basis β_1 for W consisting of eigenvectors of T_W and U_W, which are eigenvectors of T and U too.

Also, by Exercise 7 (b) of Section 6.4, we see that $W^\perp$ is T^*-invariant and U^*-invariant. But T and U are self-adjoint, so $W^\perp$ is T-invariant and U-invariant. Again, by the induction hypothesis, we may choose an orthonormal basis β_2 for $W^\perp$ consisting of eigenvectors of T and U.

Let $\beta = \beta_1 \cup \beta_2$. As V is of finite dimension, we see that β is a basis for V consisting of eigenvectors of T and U. Hence, the statement is also true for $n = k$.