Solution to Homework 7

Sec. 6.1

20. Recall from Exercise 19 (a) that

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\mathrm{Re}\langle x, y\rangle + \|y\|^2 .$$

(a) Now that $\mathbb{F} = \mathbb{R}$, so $\mathrm{Re}\langle x, y\rangle = \langle x, y\rangle$.

$$\begin{aligned}
\text{RHS} &= \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2 \\
&= \frac{1}{4}\left(\|x\|^2 + 2\mathrm{Re}\langle x, y\rangle + \|y\|^2\right) - \frac{1}{4}\left(\|x\|^2 - 2\mathrm{Re}\langle x, y\rangle + \|y\|^2\right) \\
&= \frac{1}{4}\left(4\mathrm{Re}\langle x, y\rangle\right) \\
&= \langle x, y\rangle \\
&= \text{LHS}
\end{aligned}$$

(b) Consider $\left\|x + i^k y\right\|^2$ in the right hand side.

$$\begin{aligned}
\left\|x + i^k y\right\|^2 &= \|x\|^2 + 2\mathrm{Re}\left\langle x, i^k y\right\rangle + \left\|i^k y\right\|^2 \\
&= \|x\|^2 + 2\mathrm{Re}\left(\overline{i^k}\langle x, y\rangle\right) + \|y\|^2
\end{aligned}$$

Note that $\sum_{k=1}^{4} i^k = 0$.

$$\begin{aligned}
\sum_{k=1}^{4} i^k \left\|x + i^k y\right\|^2 &= \sum_{k=1}^{4}\left(i^k\|x\|^2\right) + \sum_{k=1}^{4}\left(i^k 2\mathrm{Re}\left(\overline{i^k}\langle x, y\rangle\right)\right) + \sum_{k=1}^{4}\left(i^k\|y\|^2\right) \\
&= \|x\|^2\left(\sum_{k=1}^{4} i^k\right) + \sum_{k=1}^{4}\left(i^k 2\mathrm{Re}\left(\overline{i^k}\langle x, y\rangle\right)\right) + \|y\|^2\left(\sum_{k=1}^{4} i^k\right) \\
&= 2\sum_{k=1}^{4}\left(i^k\mathrm{Re}\left(\overline{i^k}\langle x, y\rangle\right)\right)
\end{aligned}$$

By letting $\langle x, y\rangle = a + bi$, we have the following.

$$\begin{aligned}
\sum_{k=1}^{4}\left(i^k\mathrm{Re}\left(\overline{i^k}\langle x, y\rangle\right)\right) &= ((i)(b) + (-1)(-a) + (-i)(-b) + (1)(a)) \\
&= 2(a + bi) = 2\langle x, y\rangle
\end{aligned}$$

Hence, the result follows.

23. (a) Note that the standard inner product is defined as $\langle x, y\rangle = \sum_{i=1}^{n} x_i\overline{y_i}$, so we may write $\langle x, y\rangle$ as the matrix multiplication y^*x. Then we have

$$\langle x, Ay\rangle = (Ay)^*x = y^*A^*x = \langle A^*x, y\rangle .$$

(b) From (a), we see that

$$\langle Bx, y\rangle = \langle x, Ay\rangle = \langle A^*x, y\rangle .$$

But this is true for all x and y, so we have $B = A^*$.

(c) It suffices to show that $Q^*Q = QQ^* = I$. But

$$(QQ^*)_{ij} = \sum_{k=1}^{n} q_{ik}\overline{q_{kj}} = \langle q_i, q_j\rangle ,$$

where q_i is the ith column of Q and q_{ij} is the jth entry of q_i. Obviously, we see that

$$(QQ^*)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} .$$

So $QQ^* = I$. Similarly, we have $Q^*Q = I$ and, hence, $Q^* = Q^{-1}$.

(d) Let α be the standard basis for $\mathbb{F}^n$. Then $[T]_\alpha = A$ and $[U]_\alpha = A^*$. Suppose β is an orthonormal basis β for V. As in (c), we define Q to be the matrix whose columns are the vectors in β. Note that $[I]_\beta^\alpha = Q$ and $[I]_\alpha^\beta = Q^{-1} = Q^*$. So we have the following.

$$\begin{aligned} [T]_\beta^* &= \left([I]_\alpha^\beta [T]_\alpha [I]_\beta^\alpha\right)^* \\ &= (Q^*AQ)^* \\ &= Q^*A^*Q \\ &= [I]_\alpha^\beta [U]_\alpha [I]_\beta^\alpha = [U]_\beta \end{aligned}$$

Sec. 6.2

2. To perform the Gram-Schmidt process, we set $v_1 = w_1$ and do the following orthogonalization.

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j\rangle}{\|v_j\|^2} v_j \quad \text{for } k = 2, \ldots, n$$

(f) Let $w_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix}$, $w_2 = \begin{pmatrix} 3 \\ 6 \\ 3 \\ -1 \end{pmatrix}$ and $w_3 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 8 \end{pmatrix}$. Using Gram-Schmidt, we have the following.

$$v_1 = w_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 2 \end{pmatrix}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = \begin{pmatrix} -4 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

Next, we normalize the vectors.

$$u_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} \frac{1}{\sqrt{15}} \\ -\frac{2}{\sqrt{15}} \\ -\frac{1}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{pmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \begin{pmatrix} -\frac{4}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ \frac{3}{\sqrt{30}} \end{pmatrix}$$

Finally, we take the inner product of x with u_j to get the Fourier coefficients.

$$\langle x, u_1 \rangle = -\frac{3}{\sqrt{15}}$$

$$\langle x, u_2 \rangle = \frac{4}{\sqrt{10}}$$

$$\langle x, u_3 \rangle = \frac{12}{\sqrt{30}}$$

(f) Let $w_1 = \sin(t)$, $w_2 = \cos(t)$, $w_3 = 1$ and $w_4 = t$. Using Gram-Schmidt, we have the following.

$$v_1 = w_1 = \sin(t)$$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = \cos(t)$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = 1 - \frac{4\sin(t)}{\pi}$$

$$v_4 = w_4 - \frac{\langle w_4, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_4, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle w_4, v_3\rangle}{\langle v_3, v_3\rangle} v_3 = t + \frac{4\cos(t)}{\pi} - \frac{\pi}{2}$$

Next, we normalize the vectors.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{\sqrt{2}\sin(t)}{\sqrt{\pi}}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{\sqrt{2}\cos(t)}{\sqrt{\pi}}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{\pi - 4\sin(t)}{\sqrt{\pi^3 - 8\pi}}$$

$$u_4 = \frac{v_4}{\|v_4\|} = \frac{8\cos(t) + 2\pi t - \pi^2}{\sqrt{\frac{\pi^5}{3} - 32\pi}}$$

Finally, we take the inner product of x with u_j to get the Fourier coefficients.

$$\langle x, u_1\rangle = \frac{\sqrt{2}(2\pi + 2)}{\sqrt{\pi}}$$

$$\langle x, u_2\rangle = -\frac{4\sqrt{2}}{\sqrt{\pi}}$$

$$\langle x, u_3\rangle = \frac{\pi^3 + \pi^2 - 8\pi - 8}{\sqrt{\pi^3 - 8\pi}}$$

$$\langle x, u_4\rangle = \frac{\frac{\pi^4 - 48}{3} - 16}{\sqrt{\frac{\pi^5}{3} - 32\pi}}$$

4. Let $(a, b, c) \in S^\perp$, where a, b and c are in $\mathbb{C}$. Note that $(a, b, c) \perp S$.

$$\langle (a, b, c), (1, 0, i)\rangle = a - ci = 0$$
$$\langle (a, b, c), (1, 2, 1)\rangle = a + 2b + c = 0$$

Hence, $S^\perp = \operatorname{span}(\{(i, -\frac{1}{2} - \frac{i}{2}, 1)\})$.

6. Let W be a subspace of V. Note that for any vector $v \in V$, we can express v as a sum of w and u, where $w \in W$ and $u \in W^\perp$, that is, $\langle w, u \rangle = 0$. In particular, we can take $v = x$ and we have

$$x = w + u$$

for some $w \in W$ and $u \in W^\perp$. Since $x \notin W$, we know that u is not zero. By taking $y = u \neq 0$, the result follows.

$$\langle x, y \rangle = \langle w + u, u \rangle = \langle w, u \rangle + \langle u, u \rangle = \langle u, u \rangle \neq 0$$

Note that for any $v \in V$, there exists unique vectors $w \in W$ and $z \in W^\perp$ such that

$$v = w + z.$$

As $W \cap W^\perp = \{\mathbf{0}\}$, we see that V is a direct sum of W and $W^\perp$. Since v is arbitrary, the projection on W along $W^\perp$ can be defined naturally by $T(v) = w$. Then it is easy to see that $N(T) = W^\perp$. Moreover, as w and z are orthogonal, we have $\langle w, z \rangle = \langle z, w \rangle = 0$. Hence, we have

$$\|v\|^2 = \|w\|^2 + \|z\|^2 \geq \|w\|^2 = \|T(v)\|^2 .$$

13. (a) Suppose $S_0 \subset S$. For any $x \in S^\perp$, we have x to be orthogonal to all elements in S. Since S_0 is a subset of S, x will also be orthogonal to all elements in S_0. That means $x \in S_0^\perp$ and $S^\perp \subset S_0^\perp$.

    (b) For any $x \in S$, by definition, x is orthogonal to elements in $S^\perp$. But that just means x is in $\left(S^\perp\right)^\perp$. So we have $S \subset \left(S^\perp\right)^\perp$. Note that every orthogonal complement is a subspace. Also, span(S) is the smallest subspace containing S and now that $\left(S^\perp\right)^\perp$ is a subspace containing S. Hence, we have span$(S) \subset \left(S^\perp\right)^\perp$.

    (c) By similar argument, it is easy to see that $W \subset \left(W^\perp\right)^\perp$. For $x \notin W$, by Exercise 6, there is some $y \in W^\perp$ such that $\langle x, y \rangle \neq 0$, which means $x \notin \left(W^\perp\right)^\perp$. Hence, we have $W^c \subset \left(\left(S^\perp\right)^\perp\right)^c$, where U^c means the complement of U in V. In other words, we have $\left(W^\perp\right)^\perp \subset W$. Thus, $W = \left(W^\perp\right)^\perp$.

    (d) It is easy to see that for any $x \in V$, we have $x = w + z$, where $w \in W$ and $z \in W^\perp$ are unique. Moreover, $W \cap W^\perp = \{\mathbf{0}\}$ as $x \in W$ and $x \in W^\perp$ means $\langle x, x \rangle = 0$, that is $x = \mathbf{0}$. Hence, we see that $V = W \oplus W^\perp$.

14. We first show that

$$(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp.$$

For any $x \in (W_1 + W_2)^\perp$, we have

$$\langle x, w_1 + w_2 \rangle = 0$$

for any $w_1 \in W_1$ and $w_2 \in W_2$. By taking $w_1 = 0$, we have $\langle x, w_2 \rangle = 0$ for any $w_2 \in W_2$. So $x \in W_1^\perp$. Similarly, we have $x \in W_2^\perp$ and, hence,

$$(W_1 + W_2)^\perp \subset W_1^\perp \cap W_2^\perp.$$

Conversely, for any $x \in W_1^\perp \cap W_2^\perp$, we have

$$\langle x, w_1 \rangle = 0 = \langle x, w_2 \rangle$$

for any $w_1 \in W_1$ and $w_2 \in W_2$. Then we have $x \in (W_1 + W_2)^\perp$ as

$$\langle x, w_1 + w_2 \rangle = \langle x, w_1 \rangle + \langle x, w_2 \rangle = 0 + 0 = 0.$$

Hence, $W_1^\perp \cap W_2^\perp \subset (W_1 + W_2)^\perp$. Next, we show the second equality

$$(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$$

using results from Exercise 13 and the first equality. Note that, from Exercise 13, we have $W_j = \left(W_j^\perp\right)^\perp$. Hence, we have the following.

$$\begin{aligned}
(W_1 \cap W_2)^\perp &= \left(\left(W_1^\perp\right)^\perp \cap \left(W_2^\perp\right)^\perp\right)^\perp && \text{(by Exercise 13)} \\
&= \left(\left(W_1^\perp + W_2^\perp\right)^\perp\right)^\perp && \text{(by the first equality)} \\
&= W_1^\perp + W_2^\perp && \text{(by Exercise 13 again)}
\end{aligned}$$

16. (a) Let W be the subspace spanned by S, where $S = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal subset of V, so $\langle v_i, v_i \rangle = 1$. Then for any $x \in V$, we may write $x = w + z$, where $w \in W$ and $z \in W^\perp$. But for $w \in W$, we can express w using v_1, v_2, $\ldots$, v_n.

$$x = z + a_1 v_1 + a_2 v_2 + \cdots a_n v_n$$

Note that, by taking inner product of x with v_j, we have $a_j = \langle v, v_j \rangle$ as $\langle z, v_j \rangle = 0$. Then $\|x\|^2$ can be computed in the following way.

$$\begin{aligned}
\|x\|^2 &= \left\langle z + \sum_{i=1}^{n} a_i v_i, z + \sum_{j=1}^{n} a_j v_j \right\rangle \\
&= \langle z, z \rangle + \sum_{i=1}^{n} |a_i|^2 \langle v_i, v_i \rangle \\
&= \langle z, z \rangle + \sum_{i=1}^{n} |a_i|^2 \\
&\geq \sum_{i=1}^{n} |\langle v, v_j \rangle|^2
\end{aligned}$$

(b) From the above argument, we see that the equality holds if and only if $\langle z, z \rangle = 0$ for any x in V. But this is true if and only if $z = 0$, which means $x = w + z = w \in W$, $x \in \text{span}(S)$.