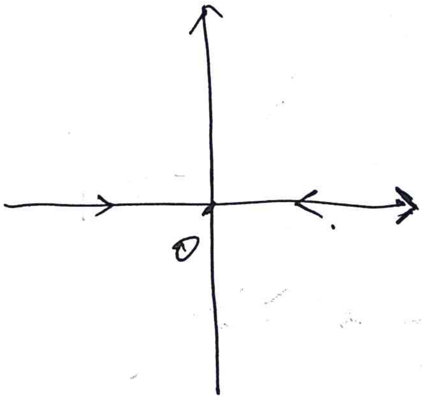
functions with no limit at one point

1. $f(x,y) = -\dfrac{x}{\sqrt{x^2+y^2}}$, $(x,y)\to(0,0)$



~~treat~~. see. x axis direction.

$(x,0)$. $= \lim_{x\to 0} -\dfrac{x}{\sqrt{x^2+0}}$

$= \lim_{x\to 0} -\dfrac{x}{|x|}$.

function $-\frac{x}{|x|}$ doesn't have limit at $x=0$.

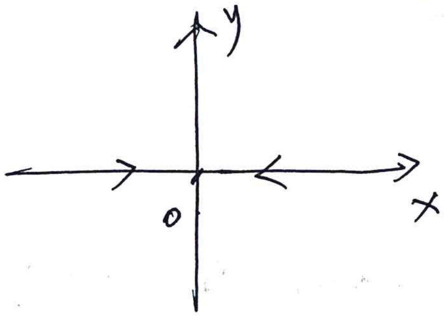
$$-\frac{x}{|x|} = \begin{cases} 1, & x<0 \\ -1, & x>0 \end{cases}$$

$\lim_{x\to 0^-} -\dfrac{x}{|x|} = 1$

$\lim_{x\to 0^+} -\dfrac{x}{|x|} = -1$.

2. $f(x,y) = \dfrac{x^4}{x^4+y^2}$, $(x,y)\to(0,0)$

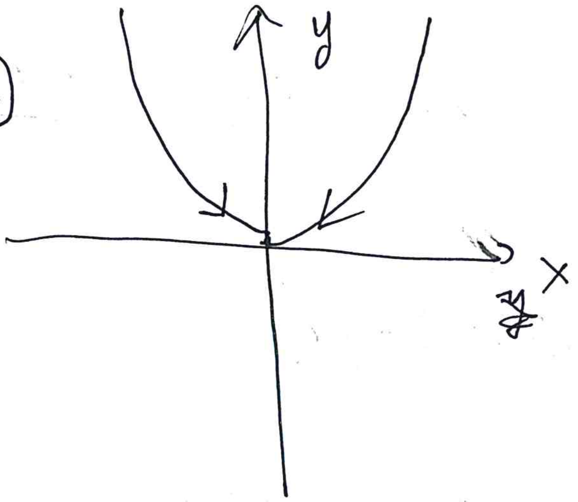
①



from. x axis. $(x,0)$

$\lim_{x\to 0} \dfrac{x^4}{x^4+0^2} = 1$.

②



from. $y = x^2$

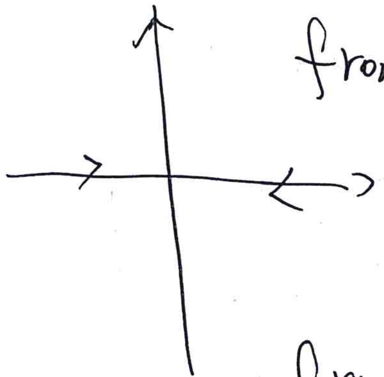
$$\lim_{x \to 0} \frac{x^4}{x^4 + (x^2)^2}$$

$$= \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \neq 1.$$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.

3. $f(x,y) = \dfrac{x^4 - y^2}{x^4 + y^2}$, $(x,y) \to (0,0)$

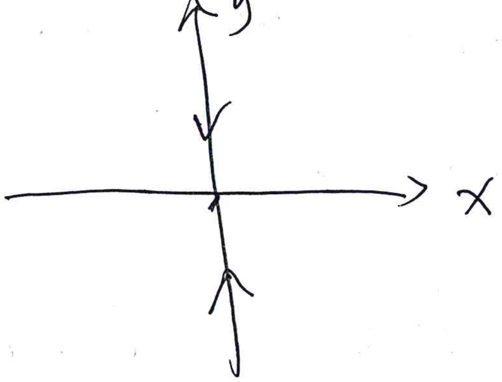
①



from x axis : $\lim\limits_{x \to 0} \dfrac{x^4 - 0^2}{x^4 + 0^2}$ (A)

$= 1$

②

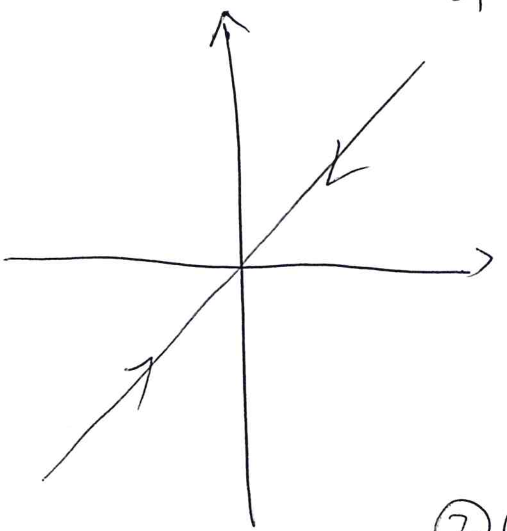


from y axis. $\lim\limits_{y \to 0} \dfrac{0^4 - y^2}{0^4 + y^2} = -1.$ (B)

$A \neq B$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)}$ doesn't exist

4. $f(x,y)=\dfrac{xy}{|xy|}$, $(x,y)\to(0,0)$

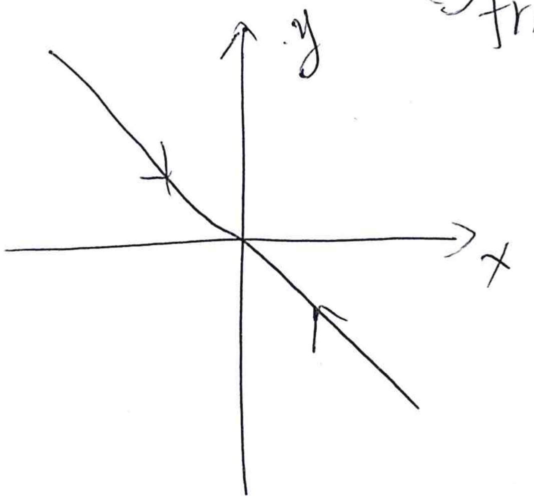


① from $y=x$.

$$\lim_{x\to 0}\frac{x^2}{x^2}=1.$$

$\neq$

② from $y=-x$. $\displaystyle\lim_{x\to 0}\frac{-x^2}{x^2}=-1$



$\Downarrow$

$\displaystyle\lim_{(y,x)\to(0,0)} f$ does not exist.

5. $f(x,y)=\dfrac{x^2+y}{y}$, $(x,y)\to(0,0)$.

① $y=x$. $\displaystyle\lim_{x\to 0}\frac{x^2+x}{x}=1.$

$\neq$

② $y=x^2$, $\displaystyle\lim_{x\to 0}\frac{x^2+x^2}{x^2}=2$ $\Rightarrow \nexists$.

($\exists$ mean exist)

6. $f(x,y)=\dfrac{x^2y}{x^4+y^2}$

① y axis $\Rightarrow$ $\displaystyle\lim_{y\to 0} f=0.$

② $y=x^2$, $\lim_{x\to 0} \frac{x^2 \cdot x^2}{x^4+x^4} = \frac{1}{2}$

$\lim$ ① $\neq$ ② $\Rightarrow$ $\nexists \lim_{(x,y)\to(0,0)} f(x,y)$

Continuous extension

Define $f(0,0)$ in a way that extends

$f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$ to be continuous at the origin.

to be defined

in order the form $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

we only need to calculate $\lim_{(x,y)\to(0,0)} f(x,y)$

we use the polar coodinate $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

treat with $|f(x,y)| = \left| xy\frac{x^2-y^2}{x^2+y^2} \right|$

after transformation.

$$|f(x,y)| = |f(r\cos\theta, r\sin\theta)|$$

$$= \left| r^2\cos\theta\cdot\sin\theta \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2} \right|$$

$$= \left| \frac{1}{2} r^2 \cdot \sin 2\theta \cdot \cos 2\theta \right|$$

$$= \left| \frac{1}{4} \cdot r^2 \sin 4\theta \right|$$

as

$$0 \le \left| \frac{1}{4} r^2 \sin 4\theta \right| \le \frac{1}{4} r^2 \to 0$$

$$(r \to 0)$$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$. we can define

$$f(0,0) = 0$$

Partial derivative

$z = xye^{-xy}$ $\rightarrow$ (treat y as constant)

$$\frac{\partial z}{\partial x} = ye^{-xy} + xye^{-xy}(-y)$$

$$= ye^{-xy} - xy^2e^{-xy}$$