Cross Product. $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$.

1.

$\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$$\vec{a} \times \vec{b} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

once $\{\vec{i}, \vec{j}, \vec{k}\}$. standard basis.

$$\vec{a} \times \vec{b} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

normalization of a vector. $\vec{v}$.

set. $\vec{v} = (v_1, v_2, v_3) \Rightarrow \|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

we call the quantity $\vec{v}_1 = \dfrac{\vec{v}}{\|v\|} = \dfrac{(v_1, v_2, v_3)}{\sqrt{v_1^2 + v_2^2 + v_3^2}}$

the normalization of $\vec{v}$

2. Geometric definition of $\vec{a} \times \vec{b}$ ~~rell~~ recall.

#

next page.

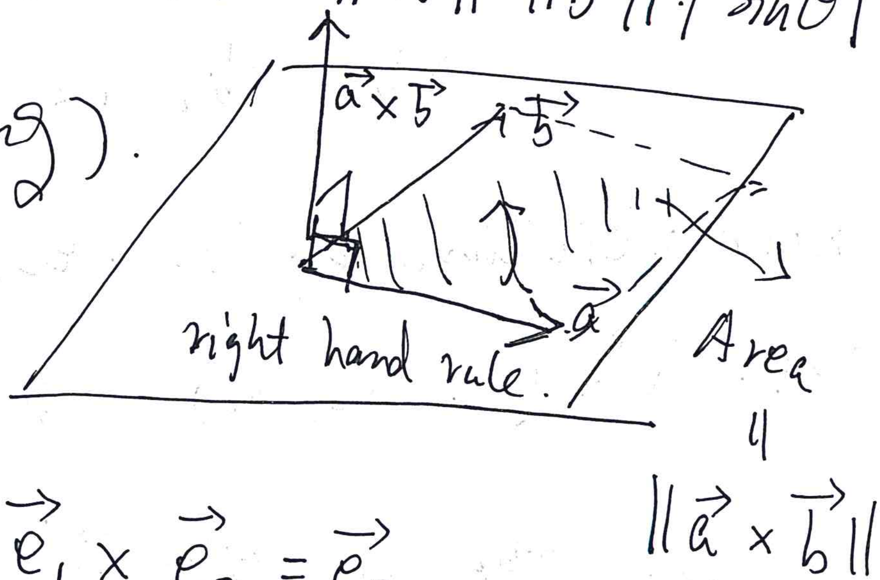
<u>thm</u>: If $\vec{a} \neq \vec{0}$, $\vec{b} \in \mathbb{R}^3$, then $\vec{a} \times \vec{b}$ is the unique vector in $\mathbb{R}^3$ s.t.

(1) $\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{a} \times \vec{b} \perp \vec{b}$

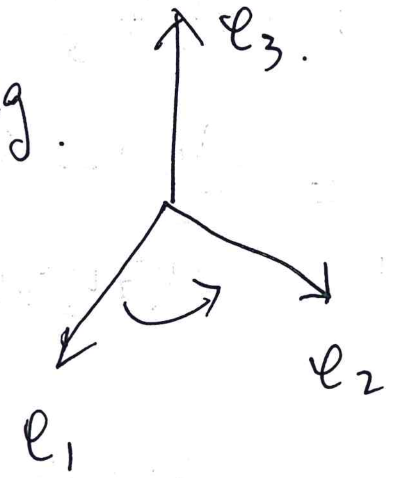
(2) $\{\vec{a}, \vec{b}, \vec{a} \times \vec{b}\}$ gives the standard orientation

(3) $\|\vec{a} \times \vec{b}\|$ = area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ $= \|\vec{a}\| \cdot \|\vec{b}\| \cdot |\sin\theta|$

(geometric meaning).



eg.



right hand system.

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$$

$$\vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$$

$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$$

<u>fact</u>: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

triple products.

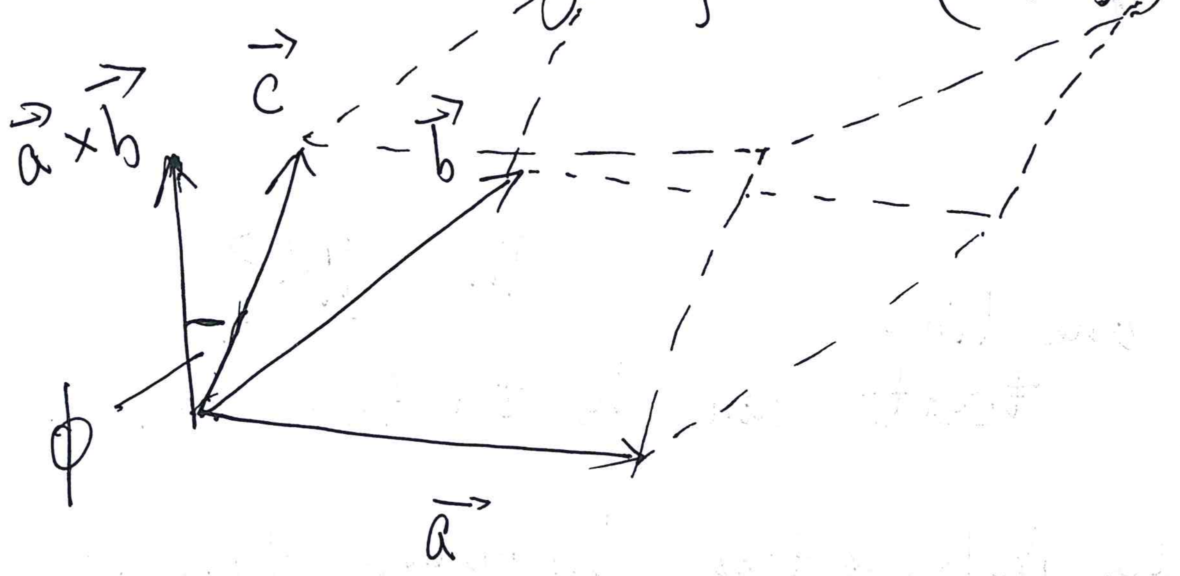
$\vec{a} \cdot (\vec{b} \times \vec{c}) \in \mathbb{R}^1$. well defined.

Prop: (1) $\vec{c} \cdot (\vec{a} \times \vec{b}) = \det \begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$.

$$= \det \begin{pmatrix} - \vec{c} - \\ - \vec{a} - \\ - \vec{b} - \end{pmatrix}$$

(2) $\vec{a} \times (\vec{b} \times \vec{c}) = \langle \vec{a}, \vec{c} \rangle \vec{b} - \langle \vec{a}, \vec{b} \rangle \vec{c}$

Geometric meaning of $\vec{c} \cdot (\vec{a} \times \vec{b})$



$$|\vec{c} \cdot (\vec{a} \times \vec{b})| = |\,\|\vec{c}\| \cdot \|\vec{a} \times \vec{b}\| \cdot \cos\phi\,|$$

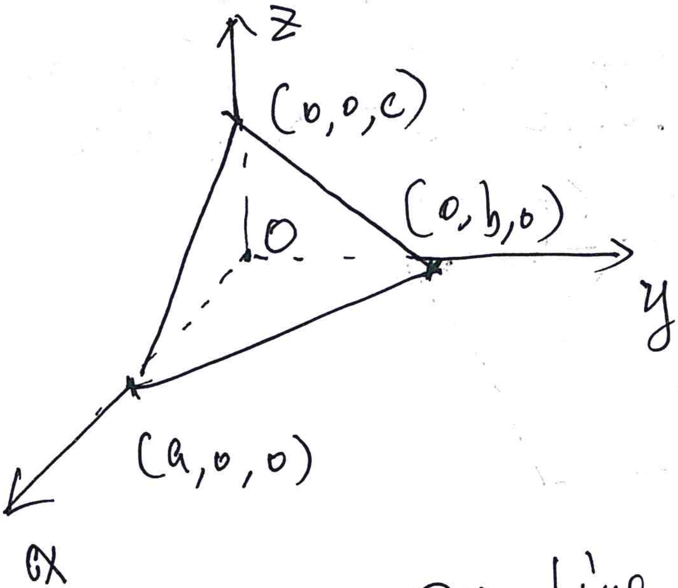
$$= |(\|\vec{c}\| \cos\phi)(\|\vec{a} \times \vec{b}\|)| = \text{vol}(\ )$$

Lines and Planes in $\mathbb{R}^3$.

Planes: form: $Ax+By+Cz+D=0$

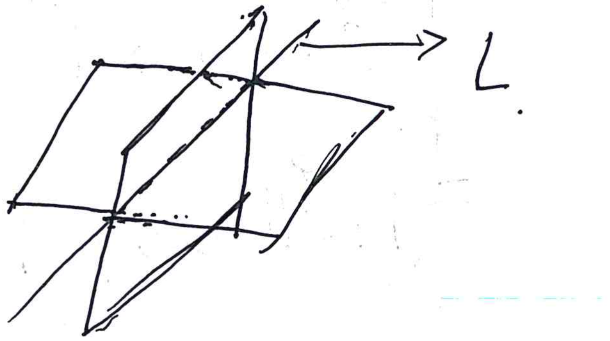
a useful form. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

$(a, b, c \neq 0)$ means a plane passing through $(a,0,0)$, $(0,b,0)$, $(0,0,c)$



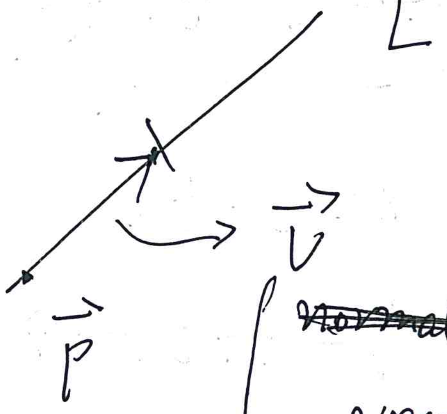
Lines: one line in $\mathbb{R}^3$ ~~treat~~ can be treated as the intersection between two planes

e.g. $L: \begin{cases} x-y+z=1 \\ 2x+y-z=0. \end{cases}$



Parametric form.

$L = \{\vec{p} + t\vec{v} \mid t \in \mathbb{R}^1\}$.



~~normalized vector.~~

$\boxed{\text{decide the direction of the line.}}$

$$\begin{cases} x - y + z = 1 & (1) \\ 2x + y - z = 0 & (2) \end{cases}$$

$(1) + (2) \Rightarrow 3x = 1 \Rightarrow x = \frac{1}{3}$

Calcl x. $\Rightarrow \begin{cases} -y + z = \frac{2}{3} \\ y - z = -\frac{2}{3} \end{cases} \Rightarrow y = z - \frac{1}{3}$.

treat z as parameter. $\underline{z = t}$

$$\Rightarrow \begin{cases} x = \frac{1}{3} \\ y = t - \frac{1}{3} \\ z = t \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0. \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

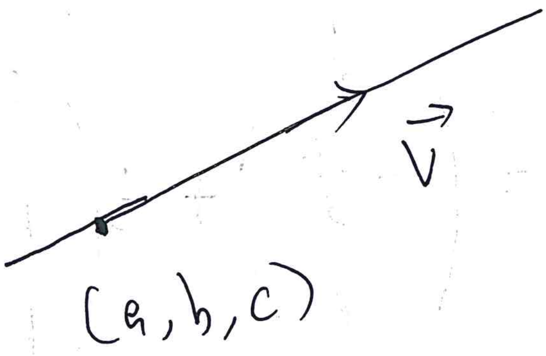
from parametric form to symmetric form

represent t. ← idea

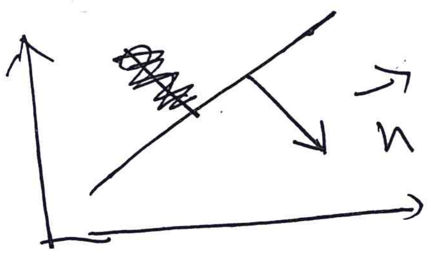
eg. $\begin{cases} x = 2 - t \\ y = 3 + 2t \\ z = 1 + t \end{cases} \Rightarrow t = \dfrac{x-2}{-1} = \dfrac{y-3}{2} = \dfrac{z-1}{1}$

generally, $\dfrac{x-a}{v_1} = \dfrac{y-b}{v_2} = \dfrac{z-c}{v_3}$.

means that a line passing through the point (a, b, c) and have the direction $\vec{v} = (v_1, v_2, v_3)$



normal vector of a line in $\mathbb{R}^2$.

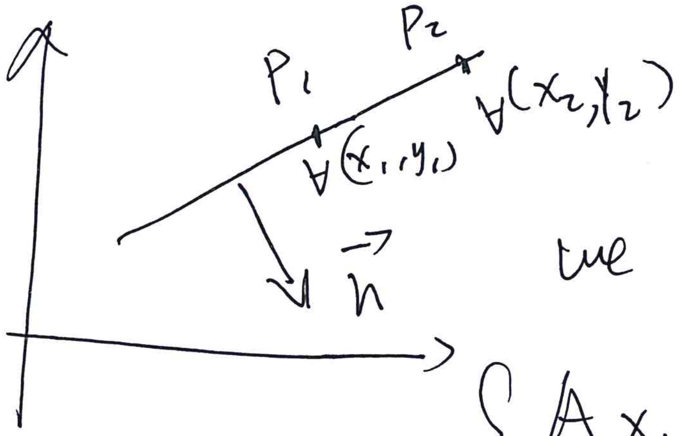


$Ax + By + C = 0$

$\vec{n} = (A, B)$

idea: set two arbitrar[y]

Point (x_1, y_1) (x_2, y_2)



we have.

$$\begin{cases} A x_1 + B y_1 + C = 0 \\ A x_2 + B y_2 + C = 0 \end{cases}$$

$$\Rightarrow \quad A(x_1 - x_2) + B(y_1 - y_2) = 0$$

namely, for $\forall$ P_1, P_2 on L.

$$(A, B) \cdot (x_1 - x_2, y_1 - y_2) = 0.$$

means $(A, B) \perp L \Rightarrow (A, B)$ is the normal vector.