MATH 2010A/B Advanced Calculus I
(2014-2015, First Term)
Homework 11
Suggested Solution

37. From the figure and the hints, the area of a triangle inscribed in a unit circle is

$$f(\alpha,\beta,\gamma)=\frac{1}{2}(1)^2\sin\alpha+\frac{1}{2}(1)^2\sin\beta+\frac{1}{2}(1)^2\sin\gamma=\frac{1}{2}(\sin\alpha+\sin\beta+\sin\gamma)$$

subjected to the constraint

$$\alpha+\beta+\gamma=2\pi$$

Let $h(\alpha,\beta,\gamma,\lambda)=f(\alpha,\beta,\gamma)+\lambda(\alpha+\beta+\gamma-2\pi)$, then

$$\frac{\partial h}{\partial\alpha}=\frac{1}{2}\cos\alpha+\lambda,\ \frac{\partial h}{\partial\beta}=\frac{1}{2}\cos\beta+\lambda,\ \frac{\partial h}{\partial\gamma}=\frac{1}{2}\cos\gamma+\lambda,\ \frac{\partial h}{\partial\lambda}=\alpha+\beta+\gamma-2\pi$$

Solve

$$\begin{cases}\frac{1}{2}\cos\alpha+\lambda & =0..........(1)\\ \frac{1}{2}\cos\beta+\lambda & =0..........(2)\\ \frac{1}{2}\cos\gamma+\lambda & =0..........(3)\\ \alpha+\beta+\gamma-2\pi & =0..........(4)\end{cases}$$

Using (1) to (3) and the fact that $0<\alpha,\beta,\gamma<\pi$, we have $\alpha=\beta=\gamma$. Then from (4), we get $\alpha=\beta=\gamma=\frac{2\pi}{3}$. Therefore, the angle at each vertex of the inscribed circle is $\dfrac{\pi}{3}$. So it is an equilateral triangle if it attains the maximum area.

38. Find maximum and minimum of

$$f(x,y)=x^2+y^2$$

subjected to

$$x^2+xy+y^2=3$$

Let $h(x,y)=x^2+xy+y^2-3$. Then $f_x=2x,\ f_y=2y,\ h_x=2x+y,\ h_y=x+2y$. Next, we need to solve

$$\begin{cases}f_x & =\lambda h_x\\ f_y & =\lambda h_y\\ h & =0\end{cases}$$

That's to say,

$$\begin{cases}2x & =\lambda(2x+y)\\ 2y & =\lambda(x+2y)\\ x^2+xy+y^2 & =3\end{cases}$$

We have

$$\begin{cases}(2-2\lambda)x-\lambda y & =0\\ -\lambda x+(2-2\lambda)y & =0\\ x^2+xy+y^2 & =3\end{cases}$$

For non-zero solution (x, y), we must have

$$\begin{aligned}(2-2\lambda)(2-2\lambda)-\lambda^2 &= 0\\ 3\lambda^2-8\lambda+4 &= 0\\ (\lambda-2)(3\lambda-2) &= 0\\ \lambda &= 2 \text{ or } \frac{2}{3}\end{aligned}$$

When $\lambda = 2$, the system becomes,

$$\begin{cases}-2x-2y &= 0\\ -2x-2y &= 0\\ x^2+xy+y^2 &= 3\end{cases}$$

From the first two equations, we know that $y = -x$. Substitute back the last equation, we got $x^2 = 3 \Rightarrow x = -\sqrt{3}$ or $\sqrt{3}$. Then $y = \sqrt{3}$ or $-\sqrt{3}$. Then $f(-\sqrt{3}, \sqrt{3}) = f(\sqrt{3}, -\sqrt{3}) = 6$.
When $\lambda = \frac{2}{3}$, the system becomes,

$$\begin{cases}2x-2y &= 0\\ -2x+2y &= 0\\ x^2+xy+y^2 &= 3\end{cases}$$

From the first two equations, we know that $y = x$. Substitute back the last equation, we got $3x^2 = 3 \Rightarrow x = -1$ or 1. Then $y = -1$ or 1. Then $f(-1, -1) = f(1, 1) = 2$.
Therefore, the closet point is $(x, y) = (-1, -1)$ or $(1, 1)$.
The farthest point is $(x, y) = (-\sqrt{3}, \sqrt{3})$ or $(\sqrt{3}, -\sqrt{3})$.

42. Find maximum and minimum of

$$f(x, y, z) = z$$

subjected to

$$z^2 = x^2 + y^2 \quad \text{and } x + 2y + 3z = 3$$

Let $h(x, y, z) = x^2 + y^2 - z^2$ and $k(x, y, z) = x + 2y + 3z - 3$.
Then $f_x = 0,\ f_y = 0,\ f_z = 1$; $h_x = 2x,\ h_y = 2y,\ h_z = -2z$; $k_x = 1,\ k_y = 2,\ k_z = 3$;
Then, we need to solve

$$\begin{cases}f_x &= \lambda h_x + \mu k_x\\ f_y &= \lambda h_y + \mu k_y\\ f_z &= \lambda h_z + \mu k_z\\ h &= 0\\ k &= 0\end{cases}$$

That is

$$\begin{cases}0 &= 2\lambda x + \mu\\ 0 &= 2\lambda y + 2\mu\\ 1 &= -2\lambda z + 3\mu\\ z^2 &= x^2 + y^2\\ x + 2y + 3z &= 3\end{cases}$$

Solving, we get $\lambda = \frac{\sqrt{5}}{6}, -\frac{\sqrt{5}}{6}; \mu = \frac{\sqrt{5}-3}{4}, \frac{-\sqrt{5}-3}{4};$

$x = \frac{9\sqrt{5}-12}{20}, \frac{-9\sqrt{5}-12}{20}; y = \frac{9\sqrt{5}-15}{10}, \frac{-9\sqrt{5}-15}{10}$ and $z = \frac{-3\sqrt{5}+9}{4}, \frac{3\sqrt{5}+9}{4}$.

Then $f(\frac{9\sqrt{5}-12}{20}, \frac{9\sqrt{5}-15}{10}, \frac{-3\sqrt{5}+9}{4}) = \frac{-3\sqrt{5}+9}{4}$, which is the lowest point.
And $f(\frac{-9\sqrt{5}-12}{20}, \frac{-9\sqrt{5}-15}{10}, \frac{3\sqrt{5}+9}{4}) = \frac{3\sqrt{5}+9}{4}$, which is the highest point.

48. Maximize

$$A = f(x, y, z, \alpha) = \frac{1}{2}xy\sin\alpha$$

subject to

$$x + y + z = P \quad \text{and} \quad z^2 = x^2 + y^2 - 2xy\cos\alpha$$

Let $h(x, y, z) = x + y + z - P$ and $k(x, y, z) = x^2 + y^2 - 2xy\cos\alpha - z^2$.

Then $f_x = \frac{1}{2}y\sin\alpha,\ f_y = \frac{1}{2}x\sin\alpha,\ f_z = 0,\ f_\alpha = \frac{1}{2}xy\cos\alpha;$
$h_x = 1,\ h_y = 1,\ h_z = 1,\ h_\alpha = 0;$
$k_x = 2x - 2y\cos\alpha,\ k_y = 2y - 2x\cos\alpha,\ k_z = -2z,\ k_\alpha = 2xy\sin\alpha;$
Then, we need to solve

$$\begin{cases} f_x &= \lambda h_x + \mu k_x \\ f_y &= \lambda h_y + \mu k_y \\ f_z &= \lambda h_z + \mu k_z \\ f_\alpha &= \lambda h_\alpha + \mu k_\alpha \\ h &= 0 \\ k &= 0 \end{cases}$$

That is,

$$\begin{cases} \frac{1}{2}y\sin\alpha &= \lambda + \mu(2x - 2y\cos\alpha) \\ \frac{1}{2}x\sin\alpha &= \lambda + \mu(2y - 2x\cos\alpha) \\ 0 &= \lambda + \mu(-2z) \\ \frac{1}{2}xy\cos\alpha &= \mu(2xy\sin\alpha) \\ x + y + z &= P \\ z^2 &= x^2 + y^2 - 2xy\cos\alpha \end{cases}$$

Solving, we will get $x = y = z$, which shows that the optimal such triangle is equilateral.

58. Note that the distance from a point (x, y, z) to the plane $2x + 3y + z = 10$ is

$$D = \left|\frac{2x + 3y + z - 10}{\sqrt{2^2 + 3^2 + 1^2}}\right| = \left|\frac{2x + 3y + z - 10}{\sqrt{14}}\right|$$

Therefore, alternatively, we need to find maximum and minimum of

$$f(x, y, z) = 14D^2 = (2x + 3y + z - 10)^2$$

subjected to

$$4x^2 + 9y^2 + z^2 = 36$$

Let $h(x, y) = 4x^2 + 9y^2 + z^2 - 36$.
Then $f_x = 4(2x + 3y + z - 10),\ f_y = 6(2x + 3y + z - 10),\ f_z = 2(2x + 3y + z - 10)$.
$h_x = 8x,\ f_y = 18y,\ f_z = 2z$.
Next, we need to solve

$$\begin{cases} f_x &= \lambda h_x \\ f_y &= \lambda h_y \\ f_z &= \lambda h_z \\ h &= 0 \end{cases}$$

That is,

$$\begin{cases} 2x + 3y + z - 10 &= 2\lambda x \\ 2x + 3y + z - 10 &= 3\lambda y \\ 2x + 3y + z - 10 &= \lambda z \\ 4x^2 + 9y^2 + z^2 &= 36 \end{cases}$$

Solving, we get $x = \pm\sqrt{3},\ y = \pm\dfrac{2\sqrt{3}}{3},\ z = \pm 2\sqrt{3}$.
$f(\sqrt{3}, \dfrac{2\sqrt{3}}{3}, 2\sqrt{3}) = (6\sqrt{3} - 10)^2$, which is the closest point.
$f(-\sqrt{3}, -\dfrac{2\sqrt{3}}{3}, -2\sqrt{3}) = (-6\sqrt{3} - 10)^2$, which is the farthest point.

60. Find maximum and minimum of

$$f(x, y, z) = z$$

subjected to

$$4x + 9y + z = 0 \quad \text{and} \quad z = 2x^2 + 3y^2$$

Let $h(x, y, z) = 4x + 9y + z$ and $k(x, y, z) = 2x^2 + 3y^2 - z$.
Then $f_x = 0,\ f_y = 0,\ f_z = 1;\ h_x = 4,\ h_y = 9,\ h_z = 1;\ k_x = 4x,\ k_y = 6y,\ k_z = -1;$
Then, we need to solve

$$\begin{cases} f_x &= \lambda h_x + \mu k_x \\ f_y &= \lambda h_y + \mu k_y \\ f_z &= \lambda h_z + \mu k_z \\ h &= 0 \\ k &= 0 \end{cases}$$

That is

$$\begin{cases} 0 &= 4\lambda + 4\mu x \\ 0 &= 9\lambda + 6\mu y \\ 1 &= \lambda - \mu \\ 4x + 9y + z &= 0 \\ z &= 2x^2 + 3y^2 \end{cases}$$

Solving, we get $\lambda = 0,\ 2; \mu = -1, 1;$

$x = 0,\ -2; y = 0,\ -3$ and $z = 0,\ 35$.

Then $f(0,0,0) = 0$, which is the lowest point.
And $f(-2,-3,-35) = 35$, which is the highest point.

62. (a) Minimize

$$f(\mathbf{x}) = x_1 + x_2 + \cdots + x_n$$

subject to the constraint

$$x_1 x_2 \cdots x_n = 1$$

Let $h(\mathbf{x}) = x_1 x_2 \cdots x_n - 1$. Then $f_{x_i} = 1$ and $h_{x_i} = x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n,\ 1 \le i \le n$
We need to solve

$$\begin{cases} f_{x_i} & = \lambda h_{x_i}, \quad 1 \le i \le n \\ h & = 0 \end{cases}$$

That is

$$\begin{cases} \lambda x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n & = 1, \quad 1 \le i \le n \\ x_1 x_2 \cdots x_n & = 1 \end{cases}$$

Multiply the first equation by x_i and use second equation, we get $\lambda = x_i, 1 \le i \le n$. Therefore, $x_1 = x_2 = \cdots = x_n$. Since x_i is positive, therefore, $x_i = 1$. So minimum value if n.

(b) If $x_i = \dfrac{a_i}{(a_1 a_2 \cdots a_n)^{1/n}}$, then $x_1 x_2 \cdots x_n = 1$. Therefore, by (a), we have

$$\begin{aligned} n &\le x_1 + x_2 + \cdots x_n \\ n &\le \frac{a_1 + a_2 + \cdots a_n}{(a_1 a_2 \cdots a_n)^{1/n}} \\ \sqrt[n]{a_1 a_2 \cdots a_n} &\le \frac{a_1 + a_2 + \cdots a_n}{n} \end{aligned}$$

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