In Problems 1 through 18, find the maximum and minimum values—if any—of the given function f subject to the given constraint or constraints.

1. $f(x, y) = 2x + y; \quad x^2 + y^2 = 1$
2. $f(x, y) = x + y; \quad x^2 + 4y^2 = 1$
3. $f(x, y) = x^2 - y^2; \quad x^2 + y^2 = 4$
4. $f(x, y) = x^2 + y^2; \quad 2x + 3y = 6$
5. $f(x, y) = xy; \quad 4x^2 + 9y^2 = 36$
6. $f(x, y) = 4x^2 + 9y^2; \quad x^2 + y^2 = 1$
7. $f(x, y, z) = x^2 + y^2 + z^2; \quad 3x + 2y + z = 6$
8. $f(x, y, z) = 3x + 2y + z; \quad x^2 + y^2 + z^2 = 1$
9. $f(x, y, z) = x + y + z; \quad x^2 + 4y^2 + 9z^2 = 36$
10. $f(x, y, z) = xyz; \quad x^2 + y^2 + z^2 = 1$
11. $f(x, y, z) = xy + 2z; \quad x^2 + y^2 + z^2 = 36$
12. $f(x, y, z) = x - y + z; \quad z = x^2 - 6xy + y^2$
13. $f(x, y, z) = x^2y^2z^2; \quad x^2 + 4y^2 + 9z^2 = 27$
14. $f(x, y, z) = x^2 + y^2 + z^2; \quad x^4 + y^4 + z^4 = 3$
15. $f(x, y, z) = x^2+y^2+z^2; \; x+y+z = 1$ and $x+2y+3z = 6$
16. $f(x, y, z) = z; \quad x^2 + y^2 = 1$ and $2x + 2y + z = 5$
17. $f(x, y, z) = z; \quad x + y + z = 1$ and $x^2 + y^2 = 1$
18. $f(x, y, z) = x; \quad x + y + z = 12$ and $4y^2 + 9z^2 = 36$
19. Find the point on the line $3x + 4y = 100$ that is closest to the origin. Use Lagrange multipliers to minimize the *square* of the distance.
20. A rectangular open-topped box is to have volume 700 in.^{3} The material for its bottom costs 7¢/in.^{2} and the material for its four vertical sides costs 5¢/in.^{2} Use the method of Lagrange multipliers to find what dimensions will minimize the cost of the material used in constructing this box.

In Problems 21 through 34, use the method of Lagrange multipliers to solve the indicated problem from Section 12.5.

21. Problem 29
22. Problem 30
23. Problem 31
24. Problem 32
25. Problem 33
26. Problem 34
27. Problem 35
28. Problem 36
29. Problem 37
30. Problem 38
31. Problem 39
32. Problem 40
33. Problem 41
34. Problem 42
35. Find the point or points of the surface $z = xy + 5$ closest to the origin. [*Suggestion:* Minimize the *square* of the distance.]
36. A triangle with sides x, y, and z has fixed perimeter $2s = x + y + z$. Its area A is given by *Heron's formula:*

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Use the method of Lagrange multipliers to show that, among all triangles with the given perimeter, the one of largest area is equilateral. [*Suggestion:* Consider maximizing A^2 rather than A.]

37. Use the method of Lagrange multipliers to show that, of all triangles inscribed in the unit circle, the one of greatest area is equilateral. [*Suggestion:* Use Fig. 12.9.9 and the fact that the area of a triangle with sides a and b and included angle θ is given by the formula $A = \frac{1}{2}ab\sin\theta$.]

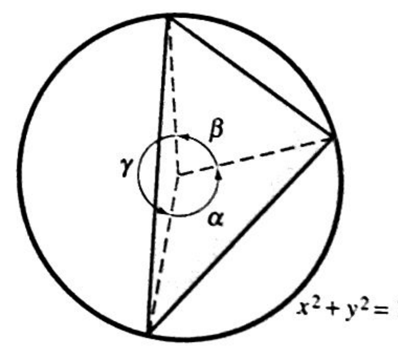

FIGURE 12.9.9 A triangle inscribed in a circle (Problem 37).

38. Find the points on the rotated ellipse $x^2 + xy + y^2 = 3$ that are closest to and farthest from the origin. [*Suggestion:* Write the Lagrange multiplier equations in the form

$$ax + by = 0,$$
$$cx + dy = 0.$$

These equations have a nontrivial solution *only if* $ad - bc = 0$. Use this fact to solve first for λ.]

39. Use the method of Problem 38 to find the points of the rotated hyperbola $x^2 + 12xy + 6y^2 = 130$ that are closest to the origin.

40. Find the points of the ellipse $4x^2 + 9y^2 = 36$ that are closest to the point $(1, 1)$ as well as the point or points farthest from it.

41. Find the highest and lowest points on the ellipse formed by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $2x + y - z = 4$.

42. [redacted] find the highest and lowest points on the ellipse formed by the intersection of the cone $z^2 = x^2 + y^2$ and the plane $x + 2y + 3z = 3$.

43. Find the points on the ellipse of Problem 42 that are nearest the origin and those that are farthest from it.

44. The ice tray shown in Fig. 12.9.10 is to be made from material that costs 1¢/in.^{2} Minimize the cost function $f(x, y, z) = xy + 3xz + 7yz$ subject to the constraints that each of the 12 compartments is to have a square horizontal cross section and that the total volume (ignoring the partitions) is to be 12 in.^{3}

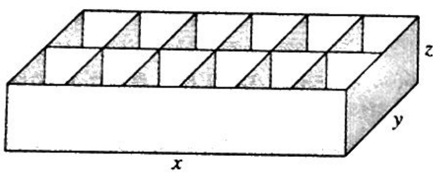

FIGURE 12.9.10 The ice tray of Problem 44.

45. Prove Theorem 1 for functions of three variables by showing that both of the vectors $\nabla f(P)$ and $\nabla g(P)$ are perpendicular at P to every curve on the surface $g(x, y, z) = 0$.

46. Find the lengths of the semiaxes of the ellipse of Example 4.

47. Figure 12.9.11 shows a right triangle with sides x, y, and z and fixed perimeter P. Maximize its area $A = \frac{1}{2}xy$ subject to the constraints $x + y + z = P$ and $x^2 + y^2 = z^2$. In particular, show that the optimal such triangle is isosceles (by showing that $x = y$).

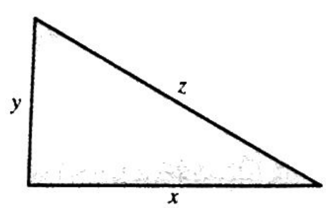

FIGURE 12.9.11 A right triangle with fixed perimeter P (Problem 47).

48. Figure 12.9.12 shows a general triangle with sides x, y, and z and fixed perimeter P. Maximize its area

$$A = f(x, y, z, \alpha) = \tfrac{1}{2}xy \sin \alpha$$

subject to the constraints $x + y + z = P$ and

$$z^2 = x^2 + y^2 - 2xy \cos \alpha$$

(the law of cosines). In particular, show that the optimal such triangle is equilateral (by showing that $x = y = z$).

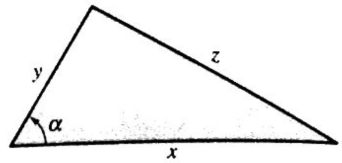

FIGURE 12.9.12 A general triangle with fixed perimeter P (Problem 48).

49. Figure 12.9.13 shows a hexagon with vertices $(0, \pm 1)$ and $(\pm x, \pm y)$ inscribed in the unit circle $x^2 + y^2 = 1$. Show that its area is maximal when it is a *regular* hexagon with equal sides and angles.

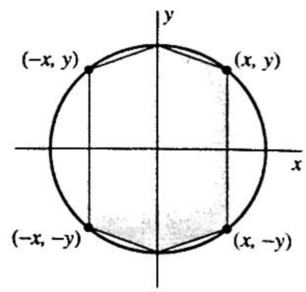

FIGURE 12.9.13 The inscribed hexagon of Problem 49.

50. When the hexagon of Fig. 12.9.13 is rotated around the y-axis, it generates a solid of revolution consisting of a cylinder and two cones (Fig. 12.9.14). What radius and cylinder height maximize the volume of this solid?

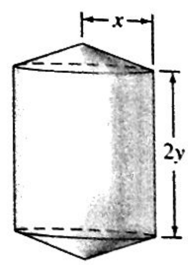

FIGURE 12.9.14 The solid of Problem 50.

In Problems 51 through 58, consider the square of the distance to be maximized or minimized. Use the numerical solution command in a computer algebra system as needed to solve the appropriate Lagrange multiplier equations.

51. Find the points of the parabola $y = (x - 1)^2$ that are closest to the origin.

52. Find the points of the ellipse $4x^2 + 9y^2 = 36$ that are closest to and farthest from the point $(3, 2)$.

53. Find the first-quadrant point of the curve $xy = 24$ that is closest to the point $(1, 4)$.

54. Find the point of the surface $xyz = 1$ that is closest to the point $(1, 2, 3)$.

55. Find the points on the sphere with center $(1, 2, 3)$ and radius 6 that are closest to and farthest from the origin.

56. Find the points of the ellipsoid $4x^2 + 9y^2 + z^2 = 36$ that are closest to and farthest from the origin.

57. Find the points of the ellipse $4x^2 + 9y^2 = 36$ that are closest to and farthest from the straight line $x + y = 10$.

58. Find the points on the ellipsoid $4x^2 + 9y^2 + z^2 = 36$ that are closest to and farthest from the plane $2x + 3y + z = 10$.

59. Find the maximum possible volume of a rectangular box that has its base in the xy-plane and its upper vertices on the elliptic paraboloid $z = 9 - x^2 - 2y^2$.

60. The plane $4x + 9y + z = 0$ intersects the elliptic paraboloid $z = 2x^2 + 3y^2$ in an ellipse. Find the highest and lowest points on this ellipse.

61. Explain carefully how the equations in (17) result from those in (15) and (16). If you wish, consider only a nontrivial special case, such as the case $n = 4$ and $k = 3$.

62. (a) Suppose that $x_1, x_2, \dots,$ and x_n are positive. Show that the minimum value of $f(\mathbf{x}) = x_1 + x_2 + \cdots + x_n$ subject to the constraint $x_1 x_2 \cdots x_n = 1$ is n. (b) Given n positive numbers $a_1, a_2, \dots, a_n$, let

$$x_i = \frac{a_i}{(a_1 a_2 \cdots a_n)^{1/n}}$$

for $1 \leqq i \leqq n$ and apply the result in part (a) to deduce the **arithmetic-geometric mean inequality**

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leqq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

63. Figure 12.9.15 shows a moat of width $a = 10$ ft, filled with alligators, and bounded on each side by a wall of height $b = 6$ ft. Soldiers plan to bridge this moat by scaling a ladder placed across the nearer wall as indicated, anchored at the ground with a handy boulder, and with the upper end directly above the far wall on the opposite side of the moat. They naturally wonder what is the minimal length L of a ladder that will suffice for this purpose. This is a particular case of the problem of minimizing the length of a line segment in the uv-plane that joins the points $P(x, 0)$ and $Q(0, y)$ on the two coordinate axes and passes through the given first-quadrant point (a, b). Show that $L_{\min} = (a^{2/3} + b^{2/3})^{3/2}$ by minimizing the squared length $f(x, y) = x^2 + y^2$ subject to the constraint that $u = a$ and $v = b$ satisfy the uv-equation $u/x + v/y = 1$ of the line through P and Q.

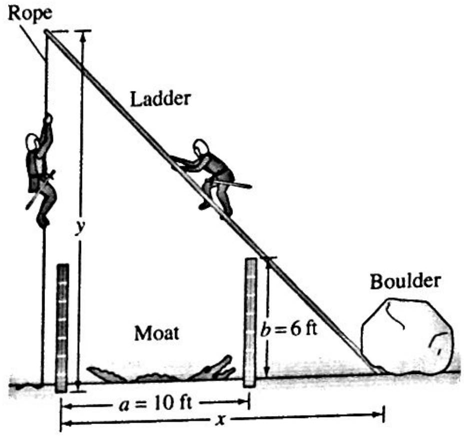

FIGURE 12.9.15 The alligator-filled moat of Problem 63.

64. A three-dimensional analog of the two-dimensional problem in Problem 63 asks for the minimal area A of the triangle in uvw-space with vertices $P(x, 0, 0)$, $Q(0, y, 0)$, and $R(0, 0, z)$ on the three coordinate axes and passing through the given first-octant point (a, b, c). (a) First deduce from Miscellaneous Problem 51 of Chapter 11 that $A^2 = \frac{1}{4}(x^2y^2 + x^2z^2 + y^2z^2)$. (b) If $a = b = c = 1$ then, by symmetry, $x = y = z$. Show in this case that $x = y = z = 3$, and thus that $A = \frac{9}{2}\sqrt{3}$. (c) Set up the Lagrange multiplier equations for minimizing the squared area A^2 subject to the constraint that the given coordinates (a, b, c) satisfy the uvw-equation $u/x + v/y + w/z = 1$ of the plane through the points P, Q, and R. In general, these equations have no known closed-form solution. Nevertheless, you can use a computer algebra system (as in the project manual for this section) to approximate numerically the minimum value of A with given numerical values of a, b, and c. Show first that with $a = b = c = 1$ you get an accurate approximation to the exact value in part (b). Then repeat the process with your own selection of values of a, b, and c. [*Note:* This three-dimensional problem was motivated by the investigation of the n-dimensional version in David Spring's article "Solution of a Calculus Problem on Minimal Volume" in *The American Mathematical Monthly* (March 2001, pp. 217–221), where a Lagrange system of $n + 1$ equations is reduced to a single nonlinear equation in a single unknown.]

65. Suppose that L_1 is the line of intersection of the planes $2x + y + 2z = 15$ and $x + 2y + 3z = 30$, and that L_2 is the line of intersection of the planes $x - y - 2z = 15$ and $3x - 2y - 3z = 20$. Find the closest points P_1 and P_2 on these two skew lines. Use a computer to solve the corresponding Lagrange multiplier system of 10 linear equations in 10 unknowns.