In Problems 1 through 4, find dw/dt both by using the chain rule and by expressing w explicitly as a function of t before differentiating.

1. $w = \exp(-x^2 - y^2)$; $x = t$, $y = \sqrt{t}$
2. $w = \dfrac{1}{u^2 + v^2}$; $u = \cos 2t$, $v = \sin 2t$
3. $w = \sin xyz$; $x = t$, $y = t^2$, $z = t^3$
4. $w = \ln(u + v + z)$; $u = \cos^2 t$, $v = \sin^2 t$, $z = t^2$

In Problems 5 through 8, find $\partial w/\partial s$ and $\partial w/\partial t$.

5. $w = \ln(x^2 + y^2 + z^2)$; $x = s - t$, $y = s + t$, $z = 2\sqrt{st}$
6. $w = pq \sin r$; $p = 2s + t$, $q = s - t$, $r = st$
7. $w = \sqrt{u^2 + v^2 + z^2}$; $u = 3e^t \sin s$, $v = 3e^t \cos s$, $z = 4e^t$
8. $w = yz + zx + xy$; $x = s^2 - t^2$, $y = s^2 + t^2$, $z = s^2t^2$

In Problems 9 through 12, find $\partial r/\partial x$, $\partial r/\partial y$, and $\partial r/\partial z$.

9. $r = e^{u+v+w}$; $u = yz$, $v = xz$, $w = xy$
10. $r = uvw - u^2 - v^2 - w^2$; $u = y + z$, $v = x + z$, $w = x + y$
11. $r = \sin(p/q)$; $p = \sqrt{xy^2z^3}$, $q = \sqrt{x + 2y + 3z}$
12. $r = \dfrac{p}{q} + \dfrac{q}{s} + \dfrac{s}{p}$; $p = e^{yz}$, $q = e^{xz}$, $s = e^{xy}$

In Problems 13 through 18, write chain rule formulas giving the partial derivative of the dependent variable p with respect to each independent variable.

13. $p = f(x, y)$; $x = x(u, v, w)$, $y = y(u, v, w)$
14. $p = f(x, y, z)$; $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$
15. $p = f(u, v, w)$; $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$
16. $p = f(v, w)$; $v = v(x, y, z, t)$, $w = w(x, y, z, t)$
17. $p = f(w)$; $w = w(x, y, z, u, v)$
18. $p = f(x, y, u, v)$; $x = x(s, t)$, $y = y(s, t)$, $u = u(s, t)$, $v = v(s, t)$

In Problems 19 through 24, find $\partial z/\partial x$ and $\partial z/\partial y$ as functions of x, y, and z, assuming that $z = f(x, y)$ satisfies the given equation.

19. $x^{2/3} + y^{2/3} + z^{2/3} = 1$
20. $x^3 + y^3 + z^3 = xyz$
21. $xe^{xy} + ye^{zx} + ze^{xy} = 3$
22. $x^5 + xy^2 + yz = 5$
23. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
24. $xyz = \sin(x + y + z)$

In Problems 25 through 28, use the method of Example 6 to find $\partial w/\partial x$ and $\partial w/\partial y$ as functions of x and y.

25. $w = u^2 + v^2 + x^2 + y^2$; $u = x - y$, $v = x + y$
26. $w = \sqrt{uvxy}$; $u = \sqrt{x - y}$, $v = \sqrt{x + y}$
27. $w = xy \ln(u + v)$; $u = (x^2 + y^2)^{1/3}$, $v = (x^3 + y^3)^{1/2}$
28. $w = uv - xy$; $u = \dfrac{x}{x^2 + y^2}$, $v = \dfrac{y}{x^2 + y^2}$

In Problems 29 through 32, write an equation for the plane tangent at the point P to the surface with the given equation.

29. $x^2 + y^2 + z^2 = 9$; $P(1, 2, 2)$
30. $x^2 + 2y^2 + 2z^2 = 14$; $P(2, 1, -2)$
31. $x^3 + y^3 + z^3 = 5xyz$; $P(2, 1, 1)$
32. $z^3 + (x + y)z^2 + x^2 + y^2 = 13$; $P(2, 2, 1)$
33. The sun is melting a rectangular block of ice. When the block's height is 1 ft and the edge of its square base is 2 ft, its height is decreasing at 2 in./h and its base edge is decreasing at 3 in./h. What is the block's rate of change of volume V at that instant?
34. A rectangular box has a square base. Find the rate at which its volume and surface area are changing if its base edge is increasing at 2 cm/min and its height is decreasing at 3 cm/min at the instant when each dimension is 1 meter.
35. Falling sand forms a conical sandpile. When the sandpile has a height of 5 ft and its base radius is 2 ft, its height is increasing at 0.4 ft/min and its base radius is increasing at 0.7 ft/min. At what rate is the volume of the sandpile increasing at that moment?
36. A rectangular block has dimensions $x = 3$ m, $y = 2$ m, and $z = 1$ m. If x and y are increasing at 1 cm/min and 2 cm/min, respectively, while z is decreasing at 2 cm/min, are the block's volume and total surface area increasing or decreasing? At what rates?
37. The volume V (in cubic centimeters) and pressure p (in atmospheres) of n moles of an ideal gas satisfy the equation $pV = nRT$, where T is its temperature (in degrees Kelvin) and $R = 82.06$. Suppose that a sample of the gas has a volume of 10 L when the pressure is 2 atm and the temperature is 300°K. If the pressure is increasing at 1 atm/min and the temperature is increasing at 10°K/min, is the volume of the gas sample increasing or decreasing? At what rate?
38. The aggregate resistance R of three variable resistances R_1, R_2, and R_3 connected in parallel satisfies the *harmonic equation*
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$
Suppose that R_1 and R_2 are 100 Ω and are increasing at 1 Ω/s, while R_3 is 200 Ω and is decreasing at 2 Ω/s. Is R increasing or decreasing at that instant? At what rate?
39. Suppose that $x = h(y, z)$ satisfies the equation $F(x, y, z) = 0$ and that $F_x \neq 0$. Show that
$$\frac{\partial x}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial x}.$$
40. Suppose that $w = f(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$. Show that
$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$
41. Suppose that $w = f(u)$ and that $u = x + y$. Show that $\partial w/\partial x = \partial w/\partial y$.

42. Suppose that $w = f(u)$ and that $u = x - y$. Show that $\partial w/\partial x = -\partial w/\partial y$ and that

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = -\frac{\partial^2 w}{\partial x \partial y}.$$

43. Suppose that $w = f(x, y)$ where $x = u + v$ and $y = u - v$. Show that

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial u\, \partial v}.$$

44. Assume that $w = f(x, y)$ where $x = 2u + v$ and $y = u - v$. Show that

$$5\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial x\, \partial y} + 2\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}.$$

45. Suppose that $w = f(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$. Show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$

[*Suggestion:* First find $\partial^2 w/\partial\theta^2$ by the method of Example 7. Then combine the result with Eqs. (7) and (8).]

46. Suppose that

$$w = \frac{1}{r} f\left(t - \frac{r}{a}\right)$$

and that $r = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{a^2}\frac{\partial^2 w}{\partial t^2}.$$

47. Suppose that $w = f(r)$ and that $r = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{d^2 w}{dr^2} + \frac{2}{r}\frac{dw}{dr}.$$

48. Suppose that $w = f(u) + g(v)$, that $u = x - at$, and that $v = x + at$. Show that

$$\frac{\partial^2 w}{\partial t^2} = a^2\frac{\partial^2 w}{\partial x^2}.$$

49. Assume that $w = f(u, v)$ where $u = x + y$ and $v = x - y$. Show that

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial u}\right)^2 - \left(\frac{\partial w}{\partial v}\right)^2.$$

50. Given: $w = f(x, y)$, $x = e^u\cos v$, and $y = e^u\sin v$. Show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2\right].$$

51. Assume that $w = f(x, y)$ and that there is a constant α such that

$$x = u\cos\alpha - v\sin\alpha \quad \text{and} \quad y = u\sin\alpha + v\cos\alpha.$$

Show that

$$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2.$$

52. Suppose that $w = f(u)$, where

$$u = \frac{x^2 - y^2}{x^2 + y^2}.$$

Show that $xw_x + yw_y = 0$.

Suppose that the equation $F(x, y, z) = 0$ defines implicitly the three functions $z = f(x, y)$, $y = g(x, z)$, and $x = h(y, z)$. To keep track of the various partial derivatives, we use the notation

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{\partial f}{\partial x}, \qquad \left(\frac{\partial z}{\partial y}\right)_x = \frac{\partial f}{\partial y}, \tag{20a}$$

$$\left(\frac{\partial y}{\partial x}\right)_z = \frac{\partial g}{\partial x}, \qquad \left(\frac{\partial y}{\partial z}\right)_x = \frac{\partial g}{\partial z}, \tag{20b}$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{\partial h}{\partial y}, \qquad \left(\frac{\partial x}{\partial z}\right)_y = \frac{\partial h}{\partial z}. \tag{20c}$$

In short, the general symbol $(\partial w/\partial u)_v$ denotes the derivative of w with respect to u, where w is regarded as a function of the independent variables u and v.

53. Using the notation in the equations in (20), show that

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

[*Suggestion:* Find the three partial derivatives on the right-hand side in terms of F_x, F_y, and F_z.]

54. Verify the result of Problem 53 for the equation

$$F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0.$$

55. Verify the result of Problem 53 (with p, V, and T in place of x, y, and z) for the equation

$$F(p, V, T) = pV - nRT = 0$$

(n and R are constants), which expresses the ideal gas law.

56. Consider a given quantity of liquid whose pressure p, volume V, and temperature T satisfy a given "state equation" of the form $F(p, V, T) = 0$. The **thermal expansivity** α and **isothermal compressivity** β of the liquid are defined by

$$\alpha = \frac{1}{V}\frac{\partial V}{\partial T} \quad \text{and} \quad \beta = -\frac{1}{V}\frac{\partial V}{\partial p}.$$

Apply Theorem 3 first to calculate $\partial V/\partial p$ and $\partial V/\partial T$, and then to calculate $\partial p/\partial V$ and $\partial p/\partial T$. Deduce from the results that $\partial p/\partial T = \alpha/\beta$.

57. The thermal expansivity and isothermal compressivity of liquid mercury are $\alpha = 1.8 \times 10^{-4}$ and $\beta = 3.9 \times 10^{-6}$, respectively, in L-atm-°C units. Suppose that a thermometer bulb is exactly filled with mercury at 50°C. If the bulb can withstand an internal pressure of no more than 200 atm, can it be heated to 55°C without breaking? *Suggestion:* Apply the result of Problem 56 to calculate the increase in pressure with each increase of one degree in temperature.

58. Suppose that the transformation $T : \mathbf{R}^3_{uvw} \to \mathbf{R}^3_{xyz}$ is defined by the functions $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$. Then its derivative matrix is defined by

$$T'(u, v, w) = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}.$$

Calculate the derivative matrix of the linear transformation defined by $x = a_1u + b_1v + c_1w$, $y = a_2u + b_2v + c_2w$, $z = a_3u + b_3v + c_3w$.