MATH 2010A/B Advanced Calculus I
(2014-2015, First Term)
Homework 6
Suggested Solution

9. $z = 3x^2 + 12x + 4y^3 - 6y^2 + 5$. $\dfrac{dz}{dx} = 6x + 12$. $\dfrac{dz}{dy} = 12y^2 - 12y$.
Solving

$$\begin{cases} 6x + 12 &= 0 \\ 12y^2 - 12y &= 0 \end{cases}$$

We have $(x, y) = (-2, 0)$, $(-2, 1)$. Then $z = -7$, -9 respectively. So $(-2, 0, -7)$, $(-2, 1, -9)$ are points at which the tangent plane is horizontal.

11. $z = (2x^2 + 3y^2)e^{-x^2-y^2}$.
$\dfrac{dz}{dx} = 4xe^{-x^2-y^2} + (2x^2 + 3y^2)e^{-x^2-y^2}(-2x) = 2x(2 - 2x^2 - 3y^2)e^{-x^2-y^2}$.
$\dfrac{dz}{dy} = 6ye^{-x^2-y^2} + (2x^2 + 3y^2)e^{-x^2-y^2}(-2y) = 2y(3 - 2x^2 - 3y^2)e^{-x^2-y^2}$.
Solving

$$\begin{cases} 2x(2 + x^2 + y^2)e^{-x^2-y^2} &= 0 \\ 2y(3 + x^2 + y^2)e^{-x^2-y^2} &= 0 \end{cases}$$

We have $(x, y) = (0, 0)$, $(0, 1)$, $(0, -1)$, $(-1, 0)$, $(1, 0)$. Then $z = 0, 3e^{-1}$, $3e^{-1}$, $2e^{-1}$, $2e^{-1}$,. Therefore, $(0, 0, 0)$, $(0, 1, 3e^{-1})$, $(0, -1, 3e^{-1})$, $(-1, 0, 2e^{-1})$, $(1, 0, 2e^{-1})$ are the points at which the tangent plane is horizontal.

13. $z = x^2 - 2x + y^2 - 2y + 3$.
$z_x = 2x - 2$, $z_y = 2y - 2$, $z_{xx} = 2$, $z_{yy} = 2$, $z_{xy} = 0$.
Solving

$$\begin{cases} 2x - 2 &= 0 \\ 2y - 2 &= 0 \end{cases}$$

We get the point $(x, y) = (1, 1)$. Then $z = 1$.

Method 1: Note that x^2 and y^2 dominate z, thus when $|x|, |y| \to \infty \Rightarrow z \to \infty$. Therefore, it is open upward. So $(1, 1, 1)$ is the lowest point.

Method 2: At this point, $z_{xx}(1, 1) = 2 > 0$ and $D = z_{xx}z_{yy} - z_{xy}^2 = (2)(2) - 0 = 4 > 0$. Therefore, $(1, 1, 1)$ is the local minimum point, and thus the lowest point.

16. $z = 4xy - x^4 - y^4$.
$z_x = 4y - 4x^3$, $z_y = 4x - 4y^3$, $z_{xx} = -12x^2$, $z_{yy} = -12y^2$, $z_{xy} = 4$.
Solving

$$\begin{cases} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{cases}$$

We get the points $(x, y) = (0, 0)$, $(-1, -1)$ and $(1, 1)$. Then $z = 0$, 2, 2.

Method 1: Note that $-x^4$ and $-y^4$ dominate z, thus when $|x|,|y| \to \infty \Rightarrow z \to -\infty$. Therefore, it is open downward. So $(0,0,2)$ and $(-1,-1,2)$ are the highest points.

Method 2: At $(x,y)=(0,0)$, $D(0,0)=z_{xx}(0,0)z_{yy}(0,0)-z_{xy}^2(0,0)=(0)(0)-4^2=16<0$, therefore, (0,0,0) is the saddle point.
At $(x,y)=(-1,-1)$, $z_{xx}(-1,-1)=-12<0$ and $D(-1,-1)=z_{xx}(-1,-1)z_{yy}(-1,-1)-z_{xy}^2(-1,-1)=(-12)(-12)-4^2=128>0$, therefore, $(-1,-1,2)$ is the highest point.
At $(x,y)=(1,1)$, $z_{xx}(1,1)=-12<0$ and $D(1,1)=z_{xx}(1,1)z_{yy}(1,1)-z_{xy}^2(1,1)=(-12)(-12)-4^2=128>0$, therefore, $(1,1,2)$ is also the highest point.

22. $z=(1+x^2)e^{-x^2-y^2}$.
$z_x=2xe^{-x^2-y^2}+(1+x^2)e^{-x^2-y^2}(-2x)=-2x^3e^{-x^2-y^2}$.
$z_y=-2y(x^2+1)e^{-x^2-y^2}$.
$z_{xx}=-6x^2e^{-x^2-y^2}-2x^3e^{-x^2-y^2}(-2x)=2x^2(2x^2-3)e^{-x^2-y^2}$.
$z_{yy}=-2(1+x^2)e^{-x^2-y^2}-2y(1+x^2)e^{-x^2-y^2}(-2y)=2(x^2+1)(2y^2-1)e^{-x^2-y^2}$.
$z_{xy}=-2x^3e^{-x^2-y^2}(-2y)=4x^3ye^{-x^2-y^2}$.
Solving
$$\begin{cases} z_x=-2x^3e^{-x^2-y^2} &=0\\ z_y=-2y(x^2+1)e^{-x^2-y^2} &=0\end{cases}$$
We get the point $(x,y)=(0,0)$. Then $z=1$.
Note that $e^{-x^2-y^2}$ dominates z, thus when $x^2+y^2\to\infty$, $z\to 0$. Therefore, it is open downward. So $(0,0,1)$ is the highest point.

23. $f(x,y)=x+2y$; R is the square with the vertices at $(\pm1,\pm1)$.
$f_x=1$ and $f_y=2$. Since $f_x\neq 0$ and $f_y\neq 0$, therefore no interior points can be maximum or minimum points.
Consider on the boundary,
Case 1: $x=-1$ and $-1\le y\le 1$,
Let $g(y)=f(-1,y)=-1+2y$, then $g_y=2\neq 0$, $g(-1)=-3, g(1)=1$.
Case 2: $x=1$ and $-1\le y\le 1$,
Let $g(y)=f(1,y)=1+2y$, then $g_y=2\neq 0$, $g(-1)=-1, g(1)=3$.
Case 3: $y=-1$ and $-1\le x\le 1$,
Let $h(x)=f(x,-1)=x-2$, then $h_x=1\neq 0$, $h(-1)=-3, h(1)=-1$.
Case 4: $y=1$ and $-1\le x\le 1$,
Let $h(x)=f(x,1)=x+2$, then $h_x=1\neq 0$, $h(-1)=1, h(1)=3$.
Therefore, the maximum value is 3 and minimum value is -3.

24. $f(x,y)=x^2+y^2-x$; R is the square with the vertices at $(\pm1,\pm1)$.
$f_x=2x-1$ and $f_y=2y$. Set $f_x=f_y=0$, we get $(x,y)=(\frac{1}{2},0)$, then $f(\frac{1}{2},0)=-\frac{1}{4}$
Consider on the boundary,
Case 1: $x=-1$ and $-1\le y\le 1$,
Let $g(y)=f(-1,y)=y^2+2$, then $g_y=2y$, $g_y=0\Rightarrow y=0$. Then $g(0)=2, g(-1)=3, g(1)=3$.
Case 2: $x=1$ and $-1\le y\le 1$,
Let $g(y)=f(1,y)=y^2$, then $g_y=2y$, $g_y=0\Rightarrow y=0$. Then $g(0)=0, g(-1)=1, g(1)=$

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Case 3: $y = -1$ and $-1 \le x \le 1$,
Let $h(x) = f(x, -1) = x^2 - x + 1$, then $h_x = 2x - 1$, $h_x = 0 \Rightarrow x = \frac{1}{2}$. Then $h(\frac{1}{2}) = \frac{3}{4}, h(-1) = 3, h(1) = 1$.
Case 4: $y = 1$ and $-1 \le x \le 1$,
Let $h(x) = f(x, 1) = x^2 - x + 1$, then $h_x = 2x - 1$, $h_x = 0 \Rightarrow x = \frac{1}{2}$. Then $h(\frac{1}{2}) = \frac{3}{4}, h(-1) = 3, h(1) = 1$.
Therefore, the maximum value is 3 and minimum value is $-\frac{1}{4}$.

28. $f(x, y) = xy^2$; R is the circular disk $x^2 + y^2 \le 3$.
$f_x = y^2$ and $f_y = 2xy$. Set $f_x = f_y = 0$, we get $(x, y) = (x_0, 0)$ for any $x_0^2 \le 3$, then $f(x_0, 0) = 0$.
Consider on the boundary $x^2 + y^2 = 3$, we have $y^2 = 3 - x^2, -\sqrt{3} \le x \le \sqrt{3}$,
Let $h(x) = f(x, y) = x(3 - x^2) = 3x - x^3$, then $h_x = 3 - 3x^2$, $h_x = 0 \Rightarrow x = \pm 1$. Then $h(-1) = -2, h(1) = 2$. Also, $h(-\sqrt{3}) = 0, h(\sqrt{3}) = 0$.
Therefore, the maximum value is 2 and minimum value is -2.

33. Find the first-octant point $P(x, y, z)$ on the surface $x^2y^2z = 4$ closest to the given fixed point $Q(0, 0, 0)$.
From the question, we need to minimize

$$f(x, y, z) = x^2 + y^2 + z^2$$

Subject to the constraint

$$g(x, y, z) = x^2y^2z - 4 = 0$$

Method 1: From the constraint, we get $z = \frac{4}{x^2y^2}$, therefore

$$h(x, y) = f(x, y, \frac{4}{x^2y^2}) = x^2 + y^2 + \frac{16}{x^4y^4}$$

Then, $h_x = 2x - \frac{64}{x^5y^4}$, $h_y = 2y - \frac{64}{x^4y^5}$. Solving

$$\begin{cases} 2x - \frac{64}{x^5y^4} & = 0 \\ 2y - \frac{64}{x^4y^5} & = 0 \end{cases}$$

We have $(x, y) = (\sqrt{2}, \sqrt{2})$ and $z = 1$. Note that when $|x|$, $|y| \to \infty$, $h \to \infty$. Therefore it is open upward. So $(\sqrt{2}, \sqrt{2}, 1)$ is the lowest point.

Method 2: Lagrange multiplier method:
Let $h(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$,

Then $h_x = 2x + \lambda 2xy^2 z$, $h_y = 2y + \lambda 2x^2 yz$, $h_z = 2z + \lambda x^2 y^2$, $h_\lambda = g(x,y,z)$.
We now need to solve the system,

$$\begin{cases} 2x + 2\lambda xy^2 z &= 0............(1) \\ 2y + 2\lambda x^2 yz &= 0............(2) \\ 2z + \lambda x^2 y^2 &= 0............(3) \\ x^2 y^2 z - 4 &= 0............(4) \end{cases}$$

From (4), we know that $x \neq 0$, $y \neq 0$ and $z \neq 0$. Therefore, from (3), $\lambda \neq 0$. Then the system becomes,

$$\begin{cases} 1 + \lambda y^2 z &= 0............(1) \\ 1 + \lambda x^2 z &= 0............(2) \\ 2z + \lambda x^2 y^2 &= 0............(3) \\ x^2 y^2 z - 4 &= 0............(4) \end{cases}$$

From (1) and (2), we have $1 + \lambda y^2 z = 1 + \lambda x^2 z \Rightarrow x = y$, then the system reduces to

$$\begin{cases} 1 + \lambda x^2 z &= 0............(1) \\ 2z + \lambda x^4 &= 0............(2) \\ x^4 z - 4 &= 0............(3) \end{cases}$$

From (3), $z = \dfrac{4}{x^4}$. Substitute this into (1) and (2), the system reduces to

$$\begin{cases} x^2 + 4\lambda &= 0............(1) \\ 8 + \lambda x^8 &= 0............(2) \end{cases}$$

From (1), $x^2 = -4\lambda$, substitute this into (2), $8 + 256\lambda^5 = 0 \Rightarrow \lambda = -\dfrac{1}{2}$.
Then, $x = \sqrt{2}$, $y = \sqrt{2}$, $z = 1$ is the required point.
Therefore the required point is $(\sqrt{2}, \sqrt{2}, 1)$.

56. From the question, we need to maximize

$$f(x,y,z) = xyz$$

Subject to the constraint

$$g(x,y,z) = x^2 + y^2 + z^2 - L^2 = 0$$

Method 1: Note that $f^2 = x^2 y^2 z^2$, then from the constraint, we get

$$h(x,y) = f^2 = x^2 y^2 (L^2 - x^2 - y^2) = L^2 x^2 y^2 - x^4 y^2 - x^2 y^4$$

Therefore, $h_x = 2L^2 xy^2 - 4x^3 y^2 - 2xy^4$, $h_y = 2L^2 x^2 y - 2x^4 y - 4x^2 y^3$. Solving

$$\begin{cases} 2L^2 xy^2 - 4x^3 y^2 - 2xy^4 &= 0 \\ 2L^2 x^2 y - 2x^4 y - 4x^2 y^3 &= 0 \end{cases}$$

We have $(x,y) = (\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}})$ and $z = \frac{L}{\sqrt{3}}$. Note that when $|x|$, $|y| \to \infty$, $h \to -\infty$. Therefore it is open downward. So $(x,y,z) = (\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}})$ is the highest point.

Method 2: Lagrange multiplier method:
Let $h(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$,
Then $h_x = yz + 2\lambda x$, $h_y = xz + 2\lambda y$, $h_z = xy + 2\lambda z$, $h_\lambda = g(x,y,z)$.
We now need to solve the system,

$$\begin{cases} yz + 2\lambda x & = 0............(1) \\ zx + 2\lambda y & = 0............(2) \\ xy + 2\lambda z & = 0............(3) \\ x^2 + y^2 + z^2 - L^2 & = 0............(4) \end{cases}$$

We then have

$$\begin{cases} xyz + 2\lambda x^2 & = 0............(1) \\ xyz + 2\lambda y^2 & = 0............(2) \\ xyz + 2\lambda z^2 & = 0............(3) \\ x^2 + y^2 + z^2 - L^2 & = 0............(4) \end{cases}$$

Note that $\lambda \neq 0$ since $x, y, z > 0$. From (1), (2) and (3), we know that $x = y = z$, substitute into (4), $x = y = z = \frac{L}{\sqrt{3}}$. Therefore, the maximum possible volume is $\frac{\sqrt{3}L^3}{9}$.

68. $f(x,y,z) = x^2 - 6xy + y^2 + 2yz + z^2 + 12$.
$f_x = 2x - 6y$, $f_y = -6x + 2y + 2z$, $f_z = 2y + 2z$.
Solving the following system,

$$\begin{cases} 2x - 6y & = 0 \\ -6x + 2y + 2z & = 0 \\ 2y + 2z & = 0 \end{cases}$$

We get $(x,y,z) = (0,0,0)$.

Method 1: $f(0,0,0) = 12$, but $f(1,1,1) = 11$ and $f(0,0,1) = 13$. Therefore, $(0,0,0)$ is a saddle point. So the function has no extrema, local or global.

Method 2: Second derivative test with Hessian Matrix:
$f_{xx} = 2$, $f_{yy} = 2$, $f_{zz} = 2$, $f_{xy} = f_{yx} = -6$, $f_{xz} = f_{zx} = 0, f_{yz} = f_{zy} = 2$. Then the Hessian matrix at $(x,y,z) = (0,0,0)$ is

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 2 & -6 & 0 \\ -6 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

Compute the eigenvalues of the Hessian matrix, we get the eigenvalues are $2-\sqrt{40}$, 2, $2+\sqrt{40}$ which has different signs. Therefore, $(0,0,0)$ is a saddle point. So the function has no extrema, local or global.

69. $f(x,y,z) = x^4 - 8x^2y^2 + y^4 + z^4 + 12$.
$f_x = 4x^3 - 16xy^2$, $f_y = 4y^3 - 16x^2y$, $f_z = 4z^3$.
Solving the following system,

$$\begin{cases} 4x^3 - 16xy^2 &= 0 \\ 4y^3 - 16x^2y &= 0 \\ 4z^3 &= 0 \end{cases}$$

We get $(x,y,z) = (0,0,0)$. Note that $f(0,0,0) = 12$, also we can compute $f(1,1,1) = 7$ and $f(1,0,0) = 13$. Therefore, $(0,0,0)$ is not a maximum or minimum point. So the function has no extrema, local or global.

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