MATH 2010A/B Advanced Calculus I

(2014-2015, First Term)

Homework 5

Suggested Solution

5. $f(x,y) = \dfrac{x+y}{x-y}$;

$f_x = \dfrac{(x-y)(1)-(1)(x+y)}{(x-y)^2} = \dfrac{-2y}{(x-y)^2}$;

$f_y = \dfrac{(x-y)(1)-(-1)(x+y)}{(x-y)^2} = \dfrac{2x}{(x-y)^2}$

6. $f(x,y) = \dfrac{xy}{x^2+y^2}$

$f_x = \dfrac{(x^2+y^2)(y)-(xy)(2x)}{(x^2+y^2)^2} = \dfrac{y(y^2-x^2)}{(x^2+y^2)^2}$;

$f_y = \dfrac{(x^2+y^2)(x)-(xy)(2y)}{(x^2+y^2)^2} = \dfrac{x(x^2-y^2)}{(x^2+y^2)^2}$;

15. $f(x,y) = x^2 e^y \ln z$;

$f_x = 2xe^y \ln z$; $f_z = x^2 e^y \ln z$; $f_z = \dfrac{x^2 e^y}{z}$;

16. $f(u,v) = (2u^2+3v^2)e^{-u^2-v^2}$;

$f_u = 4ue^{-u^2-v^2} + (2u^2+3v^2)e^{-u^2-v^2}(-2u) = 2u(2-2u^2-3v^2)e^{-u^2-v^2}$;

$f_v = 6ve^{-u^2-v^2} + (2u^2+3v^2)e^{-u^2-v^2}(-2v) = 2v(3-2u^2-3v^2)e^{-u^2-v^2}$;

18. $f(u,v) = e^{uv}(\cos uv + \sin uv)$;

$f_u = ve^{uv}(\cos uv + \sin uv) + e^{uv}(-v\sin uv + v\cos uv) = 2ve^{uv}\cos uv$;

$f_v = ue^{uv}(\cos uv + \sin uv) + e^{uv}(-u\sin uv + u\cos uv) = 2ue^{uv}\cos uv$;

19. $f(u,v,w) = ue^v + ve^w + we^u$;

$f_u = e^v + we^u$;

$f_v = ue^v + e^w$;

$f_w = ve^w + e^u$;

27. $z = e^{-3x}\cos y$;

$z_x = -3e^{-3x}\cos y \Rightarrow z_{xy} = 3e^{-3x}\sin y$;

$z_y = -e^{-3x}\sin y \Rightarrow z_{yz} = 3e^{-3x}\sin y$;

$\therefore \quad z_{xy} = z_{yx}$

38. $z = e^{-x^2-y^2}$; $P = (0,0,1)$

$z_x = -2xe^{-x^2-y^2}$; At P, $z_x = 0$;

$z_y = -2ye^{-x^2-y^2}$; At P, $z_y = 0$;

Therefore, equation of the tangent plane is

$$z - 1 = (0)(x-0) + (0)(y-0) \Rightarrow z = 1$$

40. $z = \sqrt{x^2+y^2}$; $P = (3,-4,5)$

$z_x = x(x^2+y^2)^{-\frac{1}{2}}$; At P, $z_x = 3((3)^2+(-4)^2)^{-\frac{1}{2}} = \dfrac{3}{5}$;

$z_x = y(x^2+y^2)^{-\frac{1}{2}}$; At P, $z_x = -4((3)^2+(-4)^2)^{-\frac{1}{2}} = -\dfrac{4}{5}$;
Therefore, equation of the tangent plane is

$$z-5 = (\frac{3}{5})(x-3) + (-\frac{4}{5})(y-(-4)) \Rightarrow -3x+4y+5z=0$$

55. $u = u(x,t) = e^{-n^2kt}\sin nx$;
$\dfrac{\partial u}{\partial t} = -n^2ke^{-n^2kt}\sin nx$;
$\dfrac{\partial u}{\partial x} = ne^{-n^2kt}\cos nx \Rightarrow \dfrac{\partial^2 u}{\partial^2 x} = -n^2e^{-n^2kt}\sin nx$;
Therefore,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial^2 x}$$

56. $u(x,y,t) = e^{-[m^2+n^2]kt}\sin mx\cos ny$;
$\dfrac{\partial u}{\partial t} = -(m^2+n^2)ke^{-[m^2+n^2]kt}\sin mx\cos ny$;
$\dfrac{\partial u}{\partial x} = me^{-[m^2+n^2]kt}\cos mx\cos ny \Rightarrow \dfrac{\partial^2 u}{\partial^2 x} = -m^2e^{-[m^2+n^2]kt}\cos mx\cos ny$;
$\dfrac{\partial u}{\partial y} = -ne^{-[m^2+n^2]kt}\sin mx\sin ny \Rightarrow \dfrac{\partial^2 u}{\partial^2 y} = -n^2e^{-[m^2+n^2]kt}\sin mx\cos ny$;
Therefore,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y}\right)$$

68. $z = \ln(\cos x) - \ln(\cos y)$;
$z_x = -\tan x$; $z_y = \tan y$; $z_{xx} = -\sec^2 x$; $z_{yy} = \sec^2 y$; $z_{xy} = 0$;
Then,

$$\begin{aligned}
& (1+z_y^2)z_{xx} - zz_xz_yz_{xy} + (1+z_x^2)z_{yy} \\
= {} & (1+\tan^2 y)(-\sec^2 x) - (\ln(\cos x) - \ln(\cos y))(-\tan x)(\tan y)(0) + (1+(-\tan x)^2)\sec^2 y \\
= {} & (\sec^2 y)(-\sec^2 x) + (\sec^2 x)\sec^2 y \\
= {} & 0
\end{aligned}$$

73.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{unless } x=y=0 \\ 0 & \text{if } x=y=0 \end{cases}$$

(a) For $x=y=0$,

$$\begin{aligned} f_x &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{(h)(0)}{h^2+0^2}-0}{h} \\ &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned} f_y &= \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{(0)(h)}{0^2+h^2}-0}{h} \\ &= 0 \end{aligned}$$

From question 6, we then have

$$f_x(x,y) = \begin{cases} \dfrac{y(y^2-x^2)}{(x^2+y^2)^2} & \text{unless } x=y=0 \\ 0 & \text{if } x=y=0 \end{cases}$$

And

$$f_y(x,y) = \begin{cases} \dfrac{x(x^2-y^2)}{(x^2+y^2)^2} & \text{unless } x=y=0 \\ 0 & \text{if } x=y=0 \end{cases}$$

Therefore, f_x and f_y are defined everywhere and are continuous except at the origin.

(b) Consider along the straight line $y=2x$,

$$\lim_{(x,y)\to(0,0)} f_x = \lim_{x\to 0} \frac{2x(4x^2-x^2)}{(x^2+4x^2)^2} = \lim_{x\to 0} \frac{6x^3}{25x^4} = \lim_{x\to 0} \frac{6}{25x} = \text{ limit does not exsit}$$

And

$$\lim_{(x,y)\to(0,0)} f_y = \lim_{x\to 0} \frac{x(x^2-4x^2)}{(x^2+4x^2)^2} = \lim_{x\to 0} \frac{-3x^3}{25x^4} = \lim_{x\to 0} -\frac{3}{25x} = \text{ limit does not exsit}$$

(c) For x and y not both zero,

$$\begin{aligned} f_{xx} &= \frac{(x^2+y^2)^2(-2xy)-y(y^2-x^2)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4} \\ &= \frac{(x^2+y^2)(-2xy)-4xy(y^2-x^2)}{(x^2+y^2)^3} \\ &= \frac{-2xy(-x^2+3y^2)}{(x^2+y^2)^3} \end{aligned}$$

Similarly,

$$\begin{aligned} f_{yy} &= \frac{(x^2+y^2)^2(-2xy) - x(x^2-y^2)(2)(x^2+y^2)(2y)}{(x^2+y^2)^4} \\ &= \frac{(x^2+y^2)(-2xy) - 4xy(x^2-y^2)}{(x^2+y^2)^3} \\ &= \frac{-2xy(3x^2-y^2)}{(x^2+y^2)^3} \end{aligned}$$

And

$$\begin{aligned} f_{xy} &= \frac{(x^2+y^2)^2(3y^2-x^2) - y(y^2-x^2)(2)(x^2+y^2)(2y)}{(x^2+y^2)^4} \\ &= \frac{(x^2+y^2)(3y^2-x^2) - 4y^2(y^2-x^2)}{(x^2+y^2)^3} \\ &= \frac{-x^4+6x^2y^2-y^4}{(x^2+y^2)^3} \\ f_{yx} &= \frac{(x^2+y^2)^2(3x^2-y^2) - x(x^2-y^2)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4} \\ &= \frac{(x^2+y^2)(3x^2-y^2) - 4x^2(x^2-y^2)}{(x^2+y^2)^3} \\ &= \frac{-x^4+6x^2y^2-y^4}{(x^2+y^2)^3} \end{aligned}$$

Therefore, the second-order partial derivatives of f are all defined and continuous except possible at the origin.

(d) For $x = y = 0$,

$$f_{xx} = \lim_{h\to 0} \frac{f_x(h,0) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(0)(0^2-h^2)}{(h^2+0^2)^2} - 0}{h} = 0$$

$$f_{yy} = \lim_{h\to 0} \frac{f_y(0,h) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(0)(0^2-h^2)}{(0^2+h^2)^2} - 0}{h} = 0$$

Therefore, f_{xx} and f_{yy} exist at the origin.

$$f_{xy} = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(h)(h^2-0^2)}{(0^2+h^2)^2} - 0}{h} = \lim_{h\to 0} \frac{1}{h^2} = \text{ limit does not exsit}$$

And

$$f_{yx} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(h)(h^2-0^2)}{(h^2+0^2)^2} - 0}{h} = \lim_{h\to 0} \frac{1}{h^2} = \text{ limit does not exsit}$$

Therefore, the mixed partial derivatives f_{xy} and f_{yx} do not exist.

74.

$$g(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{unless } x = y = 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

(a) For x and y not both zero,

$$\begin{aligned} g_x &= \frac{(x^2+y^2)(3x^2y-y^3)-xy(x^2-y^2)(2x)}{(x^2+y^2)^2} \\ &= \frac{(x^2+y^2)(3x^2y-y^3)-2x^2y(x^2-y^2)}{(x^2+y^2)^2} \\ &= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} \end{aligned}$$

$$\begin{aligned} g_y &= \frac{(x^2+y^2)(x^3-3xy^2)-xy(x^2-y^2)(2y)}{(x^2+y^2)^2} \\ &= \frac{(x^2+y^2)(x^3-3xy^2)-2xy^2(x^2-y^2)}{(x^2+y^2)^2} \\ &= \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} \end{aligned}$$

For $x = y = 0$,

$$\begin{aligned} g_x &= \lim_{h\to 0}\frac{g(h,0)-g(0,0)}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{(h)(0)(h^2-0^2)}{h^2+0^2}-0}{h} \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} g_y &= \lim_{h\to 0}\frac{g(0,h)-g(0,0)}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{(0)(h)(0^2-h^2)}{0^2+h^2}-0}{h} \\ &= 0 \end{aligned}$$

Therefore, g_x and g_y are defined everywhere and are continuous except at the origin.

(b) Use polar coordinates $x = r\sin\theta$ and $y = r\cos\theta$,

$$
\begin{aligned}
& \lim_{(x,y)\to(0,0)} g_x \\
= & \lim_{(x,y)\to(0,0)} \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \\
= & \lim_{r\to 0} \frac{r^5\sin^4\theta\cos\theta + 4r^5\sin^2\theta\cos^3\theta - r^5\cos^5\theta}{(r^2\sin^2\theta + \cos^2\theta)^2} \\
= & \lim_{r\to 0} r(\sin^4\theta\cos\theta + 4\sin^2\theta\cos^3\theta - \cos^5\theta) \\
= & \ 0
\end{aligned}
$$

Since $|\sin^4\theta\cos\theta + 4\sin^2\theta\cos^3\theta - \cos^5\theta|$ is bounded. Similarly,

$$
\begin{aligned}
& \lim_{(x,y)\to(0,0)} g_y \\
= & \lim_{(x,y)\to(0,0)} \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \\
= & \lim_{r\to 0} \frac{r^5\sin^5\theta - 4r^5\sin^3\theta\cos^2\theta - r^5\cos\theta\sin^4\theta}{(r^2\sin^2\theta + \cos^2\theta)^2} \\
= & \lim_{r\to 0} r(\sin^5\theta - 4\sin^3\theta\cos^2\theta - \cos\theta\sin^4\theta) \\
= & \ 0
\end{aligned}
$$

Since $|\sin^5\theta - 4\sin^3\theta\cos^2\theta - \cos\theta\sin^4\theta|$ is bounded. Therefore, g_x and g_y are continuous at $(0, 0)$.

(c) For x and y not both zero,

$$
\begin{aligned}
g_{xx} &= \frac{(x^2+y^2)^2(4x^3y+8xy^3)-(x^4y+4x^2y^3-y^5)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4} \\
&= \frac{(x^2+y^2)(4x^3y+8xy^3)-4x(x^4y+4x^2y^3-y^5)}{(x^2+y^2)^3} \\
&= \frac{-4x^3y^3+12xy^5}{(x^2+y^2)^3} \\
g_{yy} &= \frac{(x^2+y^2)^2(-8x^3y-4xy^3)-(x^5-4x^3y^2-xy^4)(2)(x^2+y^2)(2y)}{(x^2+y^2)^4} \\
&= \frac{(x^2+y^2)(-8x^3y-4xy^3)-4y(x^5-4x^3y^2-xy^4)}{(x^2+y^2)^3} \\
&= \frac{-12x^5y+4x^3y^3}{(x^2+y^2)^3} \\
g_{xy} &= \frac{(x^2+y^2)^2(x^4+12x^2y^2-5y^4)-(x^4y+4x^2y^3-y^5)(2)(x^2+y^2)(2y)}{(x^2+y^2)^4} \\
&= \frac{(x^2+y^2)(x^4+12x^2y^2-5y^4)-4y(x^4y+4x^2y^3-y^5)}{(x^2+y^2)^3} \\
&= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} \\
g_{yx} &= \frac{(x^2+y^2)^2(5x^4-12x^2y^2-y^4)-(x^5-4x^3y^2-xy^4)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4} \\
&= \frac{(x^2+y^2)(5x^4-12x^2y^2-y^4)-4x(x^5-4x^3y^2-xy^4)}{(x^2+y^2)^3} \\
&= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}
\end{aligned}
$$

Therefore, the second-order partial derivatives of g are all defined and continuous except possible at the origin.

(d) For $x = y = 0$,

$$\begin{aligned}
g_{xx} &= \lim_{h\to 0} \frac{g_x(h,0) - g_x(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{(h^4)(0) + 4(h^2)(0^3) - (0^5)}{(h^2+0^2)^2} - 0}{h} \\
&= 0 \\
g_{yy} &= \lim_{h\to 0} \frac{g_y(0,h) - g_y(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{(0^5) - 4(0^3)(h^2) - (0)(h^4)}{(0^2+h^2)^2} - 0}{h} \\
&= 0 \\
g_{xy} &= \lim_{h\to 0} \frac{g_x(0,h) - g_x(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{(0)(h^4) + 4(0^2)(h^3) - (h^5)}{(h^2+0^2)^2} - 0}{h} \\
&= \lim_{h\to 0} \frac{-h}{h} \\
&= -1 \\
g_{yx} &= \lim_{h\to 0} \frac{g_y(h,0) - g_y(h,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{(h^5) - 4(h^3)(0^2) - (h)(0^4)}{(h^2+0^2)^2} - 0}{h} \\
&= \lim_{h\to 0} \frac{h}{h} \\
&= 1
\end{aligned}$$

Therefore, all four second-order partial derivatives of g exist at the origin but that $g_{xy}(0,0) \neq g_{yx}(0,0)$.

(e) Consider along the straight line $y = x$,

$$\lim_{(x,y)\to(0,0)} g_{xx} = \lim_{(x,y)\to(0,0)} \frac{-4x^3y^3 + 12xy^5}{(x^2+y^2)^2} = \lim_{x\to 0} \frac{-4x^6 + 12x^6}{x^6} = \lim_{x\to 0} 8 = 8$$

$$\lim_{(x,y)\to(0,0)} g_{yy} = \lim_{(x,y)\to(0,0)} \frac{-12x^5y + 4x^3y^3}{(x^2+y^2)^2} = \lim_{x\to 0} \frac{-12x^6 + 4x^6}{x^6} = \lim_{x\to 0} -8 = -8$$

Consider along the straight line $y = 2x$,

$$\lim_{(x,y)\to(0,0)} g_{xy} = \lim_{(x,y)\to(0,0)} \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = \lim_{x\to 0} \frac{x^6 + 36x^6 - 144x^6 - 64x^6}{x^6} = -173$$

$$\lim_{(x,y)\to(0,0)} g_{yx} = \lim_{(x,y)\to(0,0)} \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = \lim_{x\to 0} \frac{x^6 + 36x^6 - 144x^6 - 64x^6}{x^6} = -173$$

Therefore, none of the four second-order partial derivatives of g is continuous at the origin.

♠ ♡ ♣ ♢ END ♢ ♣ ♡ ♠