In Problems 1 through 20, compute the first-order partial derivatives of each function.

1. $f(x, y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$
2. $f(x, y) = x \sin y$
3. $f(x, y) = e^x(\cos y - \sin y)$
4. $f(x, y) = e^2e^{xy}$
5. $f(x, y) = \dfrac{x + y}{x - y}$
6. $f(x, y) = \dfrac{xy}{x^2 + y^2}$
7. $f(x, y) = \ln(x^2 + y^2)$
8. $f(x, y) = (x - y)^{14}$
9. $f(x, y) = x^y$
10. $f(x, y) = \tan^{-1} xy$
11. $f(x, y, z) = x^2y^3z^4$
12. $f(x, y, z) = x^2 + y^3 + z^4$
13. $f(x, y, z) = e^{xyz}$
14. $f(x, y, z) = x^4 - 16yz$
15. $f(x, y, z) = x^2e^y \ln z$
16. $f(u, v) = (2u^2 + 3v^2)\exp(-u^2 - v^2)$
17. $f(r, s) = \dfrac{r^2 - s^2}{r^2 + s^2}$
18. $f(u, v) = e^{uv}(\cos uv + \sin uv)$
19. $f(u, v, w) = ue^v + ve^w + we^u$
20. $f(r, s, t) = (1 - r^2 - s^2 - t^2)e^{-rst}$

In Problems 21 through 30, verify that $z_{xy} = z_{yx}$.

21. $z = x^2 - 4xy + 3y^2$
22. $z = 2x^3 + 5x^2y - 6y^2 + xy^4$
23. $z = x^2 \exp(-y^2)$
24. $z = xye^{-xy}$
25. $z = \ln(x + y)$
26. $z = (x^3 + y^3)^{10}$
27. $z = e^{-3x} \cos y$
28. $z = (x + y) \sec xy$
29. $z = x^2 \cosh(1/y^2)$
30. $z = \sin xy + \tan^{-1} xy$

In Problems 31 through 40, find an equation of the plane tangent to the given surface $z = f(x, y)$ at the indicated point P.

31. $z = x^2 + y^2$; $P = (3, 4, 25)$
32. $z = \sqrt{50 - x^2 - y^2}$; $P = (4, -3, 5)$
33. $z = \sin \dfrac{\pi xy}{2}$; $P = (3, 5, -1)$
34. $z = \dfrac{4}{\pi} \tan^{-1} xy$; $P = (1, 1, 1)$
35. $z = x^3 - y^3$; $P = (3, 2, 19)$
36. $z = 3x + 4y$; $P = (1, 1, 7)$
37. $z = xy$; $P = (1, -1, -1)$
38. $z = \exp(-x^2 - y^2)$; $P = (0, 0, 1)$
39. $z = x^2 - 4y^2$; $P = (5, 2, 9)$
40. $z = \sqrt{x^2 + y^2}$; $P = (3, -4, 5)$

Recall that $f_{xy} = f_{yx}$ for a function $f(x, y)$ with continuous second-order partial derivatives. In Problems 41 through 44, apply this criterion to determine whether there exists a function $f(x, y)$ having the given first-order partial derivatives. If so, try to determine a formula for such a function $f(x, y)$.

41. $f_x(x, y) = 2xy^3, \quad f_y(x, y) = 3x^2y^2$

42. $f_x(x, y) = 5xy + y^2, \quad f_y(x, y) = 3x^2 + 2xy$

43. $f_x(x, y) = \cos^2(xy), \quad f_y(x, y) = \sin^2(xy)$

44. $f_x(x, y) = \cos x \sin y, \quad f_y(x, y) = \sin x \cos y$

Figures 12.4.12 through 12.4.17 show the graphs of a certain function $f(x, y)$ and its first- and second-order partial derivatives. In Problems 45 through 50, match that function or partial derivative with its graph.

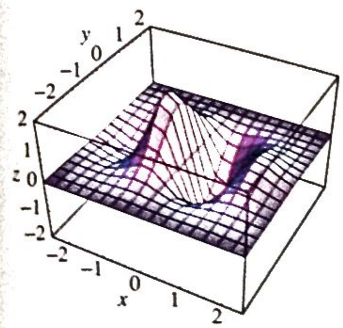

FIGURE 12.4.12

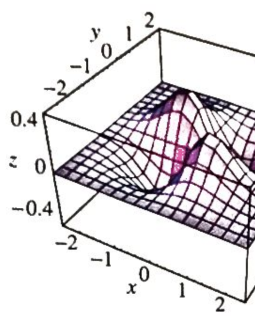

FIGURE 12.4.13

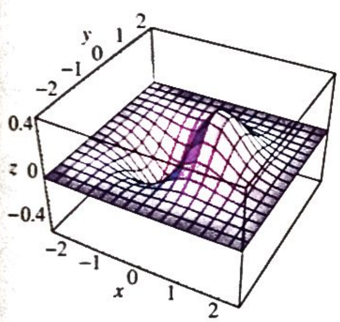

FIGURE 12.4.14

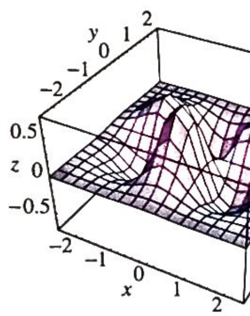

FIGURE 12.4.15

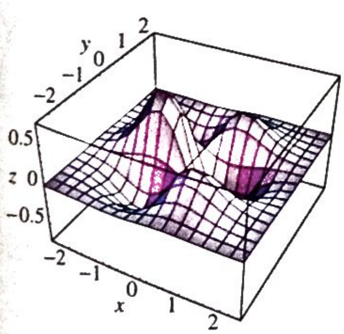

FIGURE 12.4.16

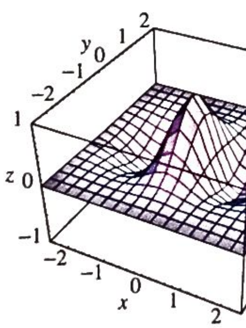

FIGURE 12.4.17

45. $f(x, y)$

46. $f_x(x, y)$

47. $f_y(x, y)$

48. $f_{xx}(x, y)$

49. $f_{xy}(x, y)$

50. $f_{yy}(x, y)$

51. Verify that the mixed second-order partial derivatives f_{xy} and f_{yx} are equal if $f(x, y) = x^m y^n$, where m and n are positive integers.

52. Suppose that $z = e^{x+y}$. Show that e^{x+y} is the result of differentiating z first m times with respect to x, then n times with respect to y.

53. Let $f(x, y, z) = e^{xyz}$. Calculate the distinct second-order partial derivatives of f and the third-order partial derivative f_{xyz}.

54. Suppose that $g(x, y) = \sin xy$. Verify that $g_{xy} = g_{yx}$ and that $g_{xxy} = g_{xyx} = g_{yxx}$.

55. It is shown in physics that the temperature $u(x, t)$ at time t at the point x of a long, insulated rod that lies along the x-axis satisfies the *one-dimensional heat equation*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad (k \text{ is a constant}).$$

Show that the function

$$u = u(x, t) = \exp(-n^2kt)\sin nx$$

satisfies the one-dimensional heat equation for any choice of the constant n.

56. The *two-dimensional heat equation* for an insulated thin plate is

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Show that the function

$$u = u(x, y, t) = \exp(-[m^2 + n^2]kt)\sin mx \cos ny$$

satisfies this equation for any choice of the constants m and n.

57. A string is stretched along the x-axis, fixed at each end, and then set into vibration. It is shown in physics that the displacement $y = y(x, t)$ of the point of the string at location x at time t satisfies the *one-dimensional wave equation*

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2},$$

where the constant a depends on the density and tension of the string. Show that the following functions satisfy the one-dimensional wave equation: (a) $y = \sin(x + at)$; (b) $y = \cosh(3[x - at])$; (c) $y = \sin kx \cos kat$ (k is a constant).

58. A steady-state temperature function $u = u(x, y)$ for a thin flat plate satisfies *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Determine which of the following functions satisfy Laplace's equation:

(a) $u = \ln\left(\sqrt{x^2 + y^2}\right)$;

(b) $u = \sqrt{x^2 + y^2}$;

(c) $u = \arctan(y/x)$;

(d) $u = e^{-x}\sin y$.

59. Suppose that f and g are twice-differentiable functions of a single variable. Show that $y(x, t) = f(x + at) + g(x - at)$ satisfies the one-dimensional wave equation of Problem 57.

60. The electric potential field of a point charge q is defined (in appropriate units) by $\phi(x, y, z) = q/r$ where $r = \sqrt{x^2 + y^2 + z^2}$. Show that ϕ satisfies the *three-dimensional Laplace equation*

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

61. Let $u(x, t)$ denote the underground temperature at depth x and time t at a location where the seasonal variation of surface ($x = 0$) temperature is described by

$$u(0, t) = T_0 + a_0 \cos \omega t,$$

where T_0 is the annual average surface temperature and the constant ω is so chosen that the period of $u(0, t)$ is one year. Show that the function

$$u(x, t) = T_0 + a_0 \exp\left(-x\sqrt{\omega/2k}\right) \cos\left(\omega t - x\sqrt{\omega/2k}\right)$$

satisfies both the "surface condition" and the one-dimensional heat equation of Problem 55.

62. The aggregate electrical resistance R of three resistances R_1, R_2, and R_3 connected in parallel satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Show that

$$\frac{\partial R}{\partial R_1} + \frac{\partial R}{\partial R_2} + \frac{\partial R}{\partial R_3} = \left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2}\right) \div \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^2.$$

63. The **ideal gas law** $pV = nRT$ (n is the number of moles of the gas, R is a constant) determines each of the three variables p (pressure), V (volume), and T (temperature) as functions of the other two. Show that

$$\frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -1.$$

64. cone $z^2 = x^2 + y^2$ passes through the origin. Show this by methods of calculus.

65. There is only one point at which the plane tangent to the surface

$$z = x^2 + 2xy + 2y^2 - 6x + 8y$$

is horizontal. Find it.

66. Show that the plane tangent to the paraboloid with equation $z = x^2 + y^2$ at the point (a, b, c) intersects the xy-plane in the line with equation $2ax + 2by = a^2 + b^2$. Then show that this line is tangent to the circle with equation $4x^2 + 4y^2 = a^2 + b^2$.

67. According to van der Waals' equation, 1 mol of a gas satisfies the equation

$$\left(p + \frac{a}{V^2}\right)(V - b) = (82.06)T$$

where p, V, and T are as in Example 4. For carbon dioxide, $a = 3.59 \times 10^6$ and $b = 42.7$, and V is 25,600 cm^3 when p is 1 atm and $T = 313$ K. (a) Compute $\partial V/\partial p$ by differentiating van der Waals' equation with T held constant. Then estimate the change in volume that would result from an increase of 0.1 atm of pressure with T held at 313 K. (b) Compute $\partial V/\partial T$ by differentiating van der Waals' equation with p held constant. Then estimate the change in volume that would result from an increase of 1 K in temperature with p held at 1 atm.

68. A *minimal surface* has the least surface area of all surfaces with the same boundary. Figure 12.4.18 shows *Scherk's minimal surface.* It has the equation

$$z = \ln(\cos x) - \ln(\cos y).$$

A minimal surface $z = f(x, y)$ is known to satisfy the partial differential equation

$$(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy} = 0.$$

Verify this in the case of Scherk's minimal surface.

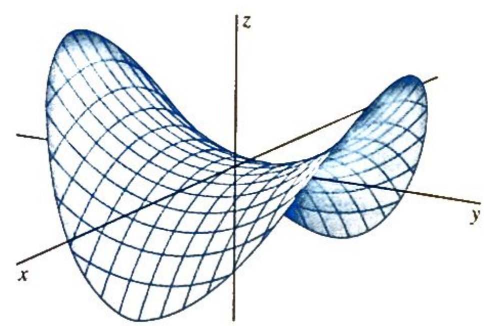

FIGURE 12.4.18 Scherk's minimal surface (Problem 68).

69. We say that the function $z = f(x, y)$ is **harmonic** if it satisfies Laplace's equation $z_{xx} + z_{yy} = 0$. (See Problem 58.) Show that each of these four functions is harmonic:

(a) $f_1(x, y) = \sin x \sinh(\pi - y)$;
(b) $f_2(x, y) = \sinh 2x \sin 2y$;
(c) $f_3(x, y) = \sin 3x \sinh 3y$;
(d) $f_4(x, y) = \sinh 4(\pi - x) \sin 4y$.

70. Figure 12.4.19 shows the graph of the sum

$$z(x, y) = \sum_{i=1}^{4} f_i(x, y)$$

of the four functions defined in Problem 69. Explain why $z(x, y)$ is a harmonic function.

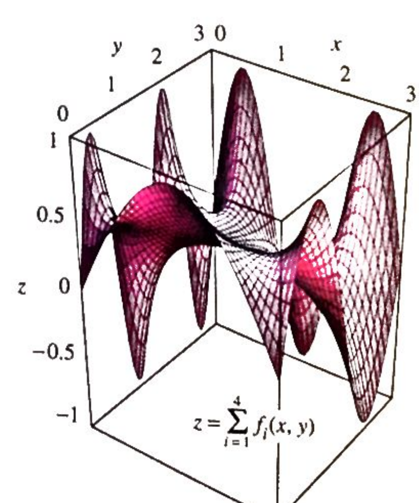

FIGURE 12.4.19 The surface $z = f(x, y)$ of Problem 70.

71. You are standing at the point where $x = y = 100$ (ft) on a hillside whose height (in feet above sea level) is given by

$$z = 100 + \frac{1}{100}(x^2 - 3xy + 2y^2),$$

with the positive x-axis to the east and the positive y-axis to the north. (a) If you head due east, will you initially be ascending or descending? At what angle (in degrees) from the horizontal? (b) If you head due north, will you initially be ascending or descending? At what angle (in degrees) from the horizontal?

72. Answer questions (a) and (b) in Problem 71, except that now you are standing at the point where $x = 150$ and $y = 250$ (ft) on a hillside whose height (in feet above sea level) is given by

$$z = 1000 + \frac{1}{1000}(3x^2 - 5xy + y^2).$$

73. Figure 12.3.7 shows the graph of the function f defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{unless } x = y = 0, \\ 0 & \text{if } x = y = 0. \end{cases}$$

(a) Show that the first-order partial derivatives f_x and f_y are defined everywhere and are continuous except possibly at the origin. (b) Consider behavior on straight lines to show that neither f_x nor f_y is continuous at the origin. (c) Show that the second-order partial derivatives of f are all defined and continuous except possibly at the origin. (d) Show that the second-order partial derivatives f_{xx} and f_{yy} exist at the origin, but that the mixed partial derivatives f_{xy} and f_{yx} do not.

74. Figure 12.4.20 shows the graph of the function g defined by

$$g(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{unless } x = y = 0, \\ 0 & \text{if } x = y = 0. \end{cases}$$

(a) Show that the first-order partial derivatives g_x and g_y are defined everywhere and are continuous except possibly at the origin. (b) Use polar coordinates to show that g_x and g_y are continuous at $(0, 0)$ as well. (c) Show that the second-order partial derivatives of g are all defined and continuous except possibly at the origin. (d) Show that all four second-order partial derivatives of g exist at the origin, but that $g_{xy}(0, 0) \neq g_{yx}(0, 0)$. (e) Consider behavior on straight lines to show that none of the four second-order partial derivatives of g is continuous at the origin.

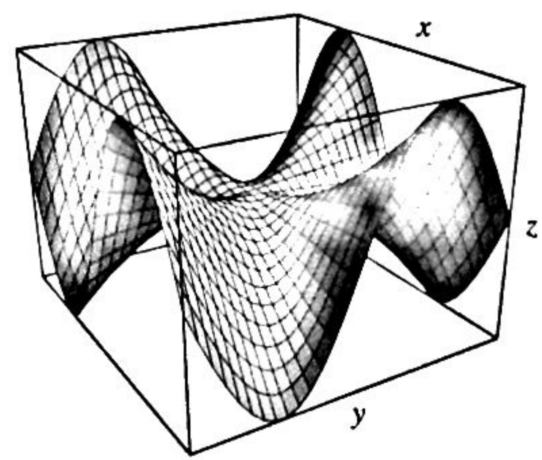

FIGURE 12.4.20 The graph $z = \dfrac{x^3y - xy^3}{x^2 + y^2}$ of Problem 74.