Exercises 14.1

Domain, Range, and Level Curves

In Exercises 1–4, find the specific function values.

1. $f(x, y) = x^2 + xy^3$
   a. $f(0, 0)$ b. $f(-1, 1)$
   c. $f(2, 3)$ d. $f(-3, -2)$
2. $f(x, y) = \sin(xy)$
   a. $f\left(2, \frac{\pi}{6}\right)$ b. $f\left(-3, \frac{\pi}{12}\right)$
   c. $f\left(\pi, \frac{1}{4}\right)$ d. $f\left(-\frac{\pi}{2}, -7\right)$
3. $f(x, y, z) = \dfrac{x - y}{y^2 + z^2}$
   a. $f(3, -1, 2)$ b. $f\left(1, \frac{1}{2}, -\frac{1}{4}\right)$
   c. $f\left(0, -\frac{1}{3}, 0\right)$ d. $f(2, 2, 100)$
4. $f(x, y, z) = \sqrt{49 - x^2 - y^2 - z^2}$
   a. $f(0, 0, 0)$ b. $f(2, -3, 6)$
   c. $f(-1, 2, 3)$ d. $f\left(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}\right)$

In Exercises 5–12, find and sketch the domain for each function.

5. $f(x, y) = \sqrt{y - x - 2}$
6. $f(x, y) = \ln(x^2 + y^2 - 4)$
7. $f(x, y) = \dfrac{(x - 1)(y + 2)}{(y - x)(y - x^3)}$
8. $f(x, y) = \dfrac{\sin(xy)}{x^2 + y^2 - 25}$
9. $f(x, y) = \cos^{-1}(y - x^2)$
10. $f(x, y) = \ln(xy + x - y - 1)$
11. $f(x, y) = \sqrt{(x^2 - 4)(y^2 - 9)}$
12. $f(x, y) = \dfrac{1}{\ln(4 - x^2 - y^2)}$

In Exercises 13–16, find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c. We refer to these level curves as a contour map.

13. $f(x, y) = x + y - 1, \quad c = -3, -2, -1, 0, 1, 2, 3$
14. $f(x, y) = x^2 + y^2, \quad c = 0, 1, 4, 9, 16, 25$
15. $f(x, y) = xy, \quad c = -9, -4, -1, 0, 1, 4, 9$
16. $f(x, y) = \sqrt{25 - x^2 - y^2}, \quad c = 0, 1, 2, 3, 4$

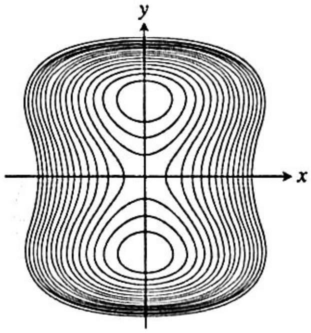
In Exercises 17–30, **(a)** find the function's domain, **(b)** find the function's range, **(c)** describe the function's level curves, **(d)** find the boundary of the function's domain, **(e)** determine if the domain is an open region, a closed region, or neither, and **(f)** decide if the domain is bounded or unbounded.

17. $f(x, y) = y - x$
18. $f(x, y) = \sqrt{y - x}$
19. $f(x, y) = 4x^2 + 9y^2$
20. $f(x, y) = x^2 - y^2$
21. $f(x, y) = xy$
22. $f(x, y) = y/x^2$
23. $f(x, y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$
24. $f(x, y) = \sqrt{9 - x^2 - y^2}$
25. $f(x, y) = \ln(x^2 + y^2)$
26. $f(x, y) = e^{-(x^2+y^2)}$
27. $f(x, y) = \sin^{-1}(y - x)$
28. $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$
29. $f(x, y) = \ln(x^2 + y^2 - 1)$
30. $f(x, y) = \ln(9 - x^2 - y^2)$

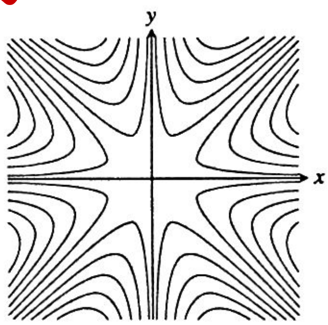
Matching Surfaces with Level Curves

Exercises 31–36 show level curves for the functions graphed in (a)–(f) on the following page. Match each set of curves with the appropriate function.

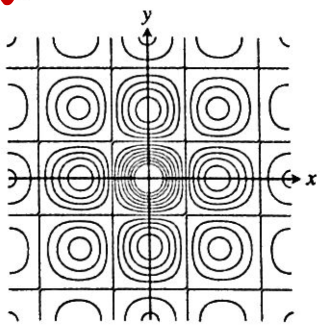
31.



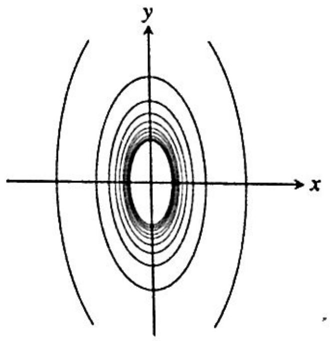
32.



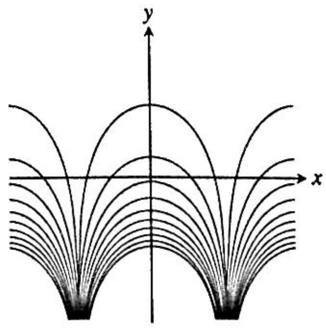
33.



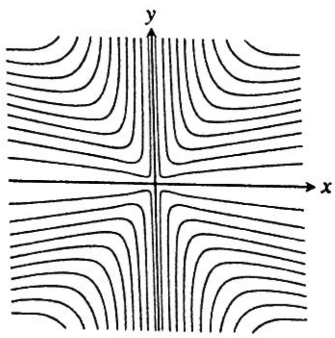
34.



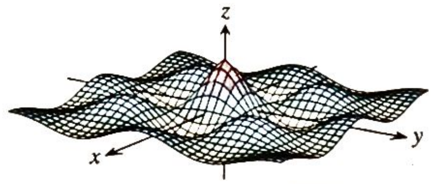
35.



36.

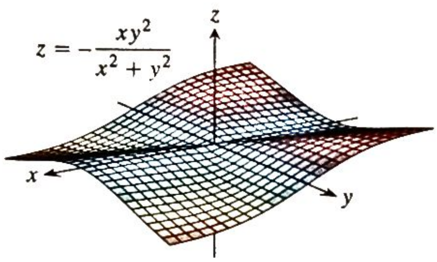


a.



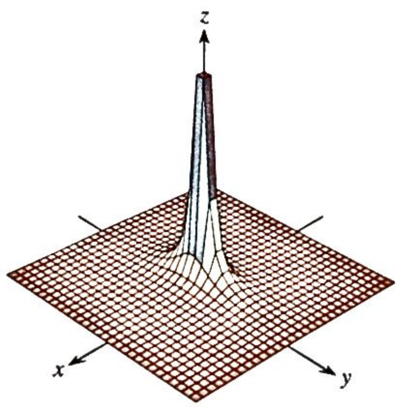
$z = (\cos x)(\cos y)\, e^{-\sqrt{x^2+y^2}/4}$

b.



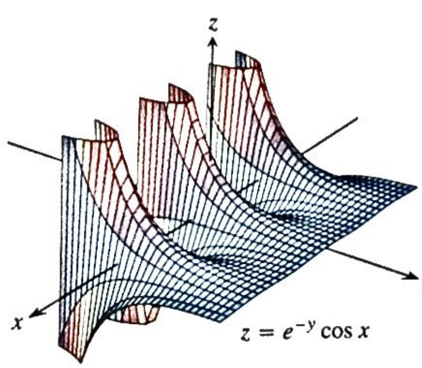
$z = -\dfrac{xy^2}{x^2+y^2}$

c.



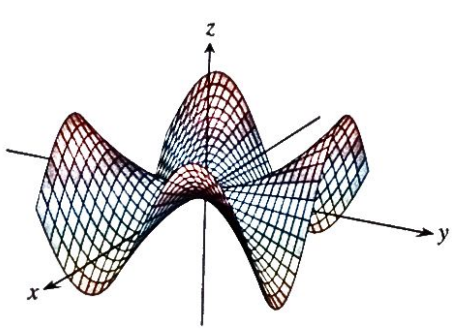
$z = \dfrac{1}{4x^2+y^2}$

d.



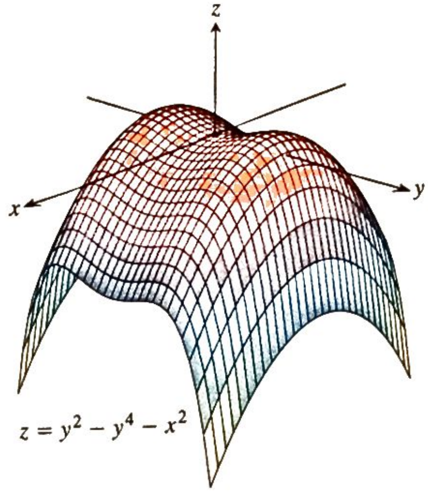
$z = e^{-y}\cos x$

e.



$z = \dfrac{xy(x^2-y^2)}{x^2+y^2}$

f.



$z = y^2 - y^4 - x^2$

Functions of Two Variables

Display the values of the functions in Exercises 37–48 in two ways: **(a)** by sketching the surface $z = f(x, y)$ and **(b)** by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.

37. $f(x, y) = y^2$

38. $f(x, y) = \sqrt{x}$

39. $f(x, y) = x^2 + y^2$

40. $f(x, y) = \sqrt{x^2 + y^2}$

41. $f(x, y) = x^2 - y$

42. $f(x, y) = 4 - x^2 - y^2$

43. $f(x, y) = 4x^2 + y^2$

44. $f(x, y) = 6 - 2x - 3y$

45. $f(x, y) = 1 - |y|$

46. $f(x, y) = 1 - |x| - |y|$

47. $f(x, y) = \sqrt{x^2 + y^2 + 4}$

48. $f(x, y) = \sqrt{x^2 + y^2 - 4}$

Finding Level Curves

In Exercises 49–52, find an equation for and sketch the graph of the level curve of the function $f(x, y)$ that passes through the given point.

49. $f(x, y) = 16 - x^2 - y^2, \quad \left(2\sqrt{2}, \sqrt{2}\right)$

50. $f(x, y) = \sqrt{x^2 - 1}, \quad (1, 0)$

51. $f(x, y) = \sqrt{x + y^2 - 3}, \quad (3, -1)$

52. $f(x, y) = \dfrac{2y - x}{x + y + 1}, \quad (-1, 1)$

Sketching Level Surfaces

In Exercises 53–60, sketch a typical level surface for the function.

53. $f(x, y, z) = x^2 + y^2 + z^2$

54. $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

55. $f(x, y, z) = x + z$

56. $f(x, y, z) = z$

57. $f(x, y, z) = x^2 + y^2$

58. $f(x, y, z) = y^2 + z^2$

59. $f(x, y, z) = z - x^2 - y^2$

60. $f(x, y, z) = (x^2/25) + (y^2/16) + (z^2/9)$

Finding Level Surfaces

In Exercises 61–64, find an equation for the level surface of the function through the given point.

61. $f(x, y, z) = \sqrt{x - y} - \ln z, \quad (3, -1, 1)$

62. $f(x, y, z) = \ln(x^2 + y + z^2), \quad (-1, 2, 1)$

63. $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad (1, -1, \sqrt{2})$

64. $g(x, y, z) = \dfrac{x - y + z}{2x + y - z}, \quad (1, 0, -2)$

In Exercises 65–68, find and sketch the domain of f. Then find an equation for the level curve or surface of the function passing through the given point.

65. $f(x, y) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n, \quad (1, 2)$

66. $g(x, y, z) = \displaystyle\sum_{n=0}^{\infty} \frac{(x + y)^n}{n!z^n}, \quad (\ln 4, \ln 9, 2)$

67. $f(x, y) = \displaystyle\int_x^y \frac{d\theta}{\sqrt{1 - \theta^2}}, \quad (0, 1)$

68. $g(x, y, z) = \displaystyle\int_x^y \frac{dt}{1 + t^2} + \int_0^z \frac{d\theta}{\sqrt{4 - \theta^2}}, \quad (0, 1, \sqrt{3})$

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for each of the functions in Exercises 69–72.

a. Plot the surface over the given rectangle.

b. Plot several level curves in the rectangle.

c. Plot the level curve of f through the given point.

69. $f(x, y) = x \sin \dfrac{y}{2} + y \sin 2x, \quad 0 \le x \le 5\pi, \quad 0 \le y \le 5\pi,$
$P(3\pi, 3\pi)$

70. $f(x, y) = (\sin x)(\cos y)e^{\sqrt{x^2+y^2}/8}, \quad 0 \le x \le 5\pi,$
$0 \le y \le 5\pi, \quad P(4\pi, 4\pi)$

71. $f(x, y) = \sin(x + 2\cos y), \quad -2\pi \le x \le 2\pi,$
$-2\pi \le y \le 2\pi, \quad P(\pi, \pi)$

72. $f(x, y) = e^{(x^{0.1} - y)} \sin(x^2 + y^2), \quad 0 \le x \le 2\pi,$
$-2\pi \le y \le \pi, \quad P(\pi, -\pi)$

Use a CAS to plot the implicitly defined level surfaces in Exercises 73–76.

73. $4 \ln(x^2 + y^2 + z^2) = 1$

74. $x^2 + z^2 = 1$

75. $x + y^2 - 3z^2 = 1$

76. $\sin\left(\dfrac{x}{2}\right) - (\cos y)\sqrt{x^2 + z^2} = 2$

Parametrized Surfaces Just as you describe curves in the plane parametrically with a pair of equations $x = f(t), y = g(t)$ defined on some parameter interval I, you can sometimes describe surfaces in space with a triple of equations $x = f(u, v), y = g(u, v), z = h(u, v)$ defined on some parameter rectangle $a \le u \le b, c \le v \le d$. Many computer algebra systems permit you to plot such surfaces in *parametric mode*. (Parametrized surfaces are discussed in detail in Section 16.5.) Use a CAS to plot the surfaces in Exercises 77–80. Also plot several level curves in the xy-plane.

77. $x = u \cos v, \quad y = u \sin v, \quad z = u, \quad 0 \le u \le 2,$
$0 \le v \le 2\pi$

78. $x = u \cos v, \quad y = u \sin v, \quad z = v, \quad 0 \le u \le 2,$
$0 \le v \le 2\pi$

79. $x = (2 + \cos u) \cos v, \quad y = (2 + \cos u) \sin v, \quad z = \sin u,$
$0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$

80. $x = 2 \cos u \cos v, \quad y = 2 \cos u \sin v, \quad z = 2 \sin u,$
$0 \le u \le 2\pi, \quad 0 \le v \le \pi$

Exercises 14.2

Limits with Two Variables

Find the limits in Exercises 1–12.

1. $\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

2. $\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}}$

3. $\lim_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$

4. $\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2$

5. $\lim_{(x,y)\to(0,\pi/4)} \sec x \tan y$

6. $\lim_{(x,y)\to(0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$

7. $\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$

8. $\lim_{(x,y)\to(1,1)} \ln|1 + x^2y^2|$

9. $\lim_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x}$

10. $\lim_{(x,y)\to(1/27,\,\pi^3)} \cos\sqrt[3]{xy}$

11. $\lim_{(x,y)\to(1,\,\pi/6)} \dfrac{x \sin y}{x^2 + 1}$

12. $\lim_{(x,y)\to(\pi/2,0)} \dfrac{\cos y + 1}{y - \sin x}$

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

13. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - 2xy + y^2}{x - y}$

14. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - y^2}{x - y}$

15. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{xy - y - 2x + 2}{x - 1}$

16. $\lim_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\, x\neq x^2}} \dfrac{y + 4}{x^2y - xy + 4x^2 - 4x}$

17. $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \dfrac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$

18. $\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \dfrac{x + y - 4}{\sqrt{x + y} - 2}$

19. $\lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \dfrac{\sqrt{2x - y} - 2}{2x - y - 4}$

20. $\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{\sqrt{x} - \sqrt{y + 1}}{x - y - 1}$

21. $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$

22. $\lim_{(x,y)\to(0,0)} \dfrac{1 - \cos(xy)}{xy}$

23. $\lim_{(x,y)\to(1,-1)} \dfrac{x^3 + y^3}{x + y}$

24. $\lim_{(x,y)\to(2,2)} \dfrac{x - y}{x^4 - y^4}$

Limits with Three Variables

Find the limits in Exercises 25–30.

25. $\lim_{P\to(1,3,4)} \left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$

26. $\lim_{P\to(1,-1,-1)} \dfrac{2xy + yz}{x^2 + z^2}$

27. $\lim_{P\to(\pi,\pi,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$

28. $\lim_{P\to(-1/4,\pi/2,2)} \tan^{-1} xyz$

29. $\lim_{P\to(\pi,0,3)} ze^{-2y}\cos 2x$

30. $\lim_{P\to(2,-3,6)} \ln\sqrt{x^2 + y^2 + z^2}$

Continuity in the Plane

At what points (x, y) in the plane are the functions in Exercises 31–34 continuous?

31. a. $f(x, y) = \sin(x + y)$ b. $f(x, y) = \ln(x^2 + y^2)$

32. a. $f(x, y) = \dfrac{x + y}{x - y}$ b. $f(x, y) = \dfrac{y}{x^2 + 1}$

33. a. $g(x, y) = \sin\dfrac{1}{xy}$ b. $g(x, y) = \dfrac{x + y}{2 + \cos x}$

34. a. $g(x, y) = \dfrac{x^2 + y^2}{x^2 - 3x + 2}$ b. $g(x, y) = \dfrac{1}{x^2 - y}$

Continuity in Space

At what points (x, y, z) in space are the functions in Exercises 35–40 continuous?

35. a. $f(x, y, z) = x^2 + y^2 - 2z^2$
 b. $f(x, y, z) = \sqrt{x^2 + y^2 - 1}$

36. a. $f(x, y, z) = \ln xyz$ b. $f(x, y, z) = e^{x+y}\cos z$

37. a. $h(x, y, z) = xy \sin\dfrac{1}{z}$ b. $h(x, y, z) = \dfrac{1}{x^2 + z^2 - 1}$

38. a. $h(x, y, z) = \dfrac{1}{|y| + |z|}$ b. $h(x, y, z) = \dfrac{1}{|xy| + |z|}$

39. a. $h(x, y, z) = \ln(z - x^2 - y^2 - 1)$
 b. $h(x, y, z) = \dfrac{1}{z - \sqrt{x^2 + y^2}}$

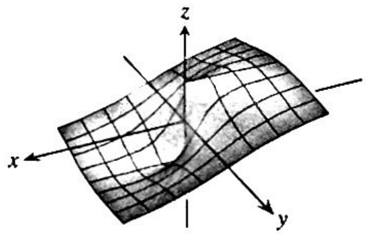
40. a. $h(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$
 b. $h(x, y, z) = \dfrac{1}{4 - \sqrt{x^2 + y^2 + z^2 - 9}}$

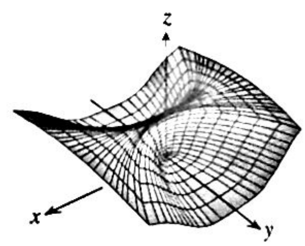
No Limit at a Point

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \to (0, 0)$.

41. $f(x, y) = -\dfrac{x}{\sqrt{x^2 + y^2}}$

42. $f(x, y) = \dfrac{x^4}{x^4 + y^2}$





43. $f(x, y) = \dfrac{x^4 - y^2}{x^4 + y^2}$

44. $f(x, y) = \dfrac{xy}{|xy|}$

45. $g(x, y) = \dfrac{x - y}{x + y}$

46. $g(x, y) = \dfrac{x^2 - y}{x - y}$

47. $h(x, y) = \dfrac{x^2 + y}{y}$

48. $h(x, y) = \dfrac{x^2y}{x^4 + y^2}$

Theory and Examples

In Exercises 49 and 50, show that the limits do not exist.

49. $\lim_{(x,y)\to(1,1)} \dfrac{xy^2 - 1}{y - 1}$

50. $\lim_{(x,y)\to(1,-1)} \dfrac{xy + 1}{x^2 - y^2}$

51. Let $f(x, y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Find each of the following limits, or explain that the limit does not exist.

a. $\lim_{(x,y)\to(0,1)} f(x, y)$

b. $\lim_{(x,y)\to(2,3)} f(x, y)$

c. $\lim_{(x,y)\to(0,0)} f(x, y)$

52. Let $f(x, y) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}$.

Find the following limits.

a. $\lim_{(x, y) \to (3, -2)} f(x, y)$

b. $\lim_{(x, y) \to (-2, 1)} f(x, y)$

c. $\lim_{(x, y) \to (0, 0)} f(x, y)$

53. Show that the function in Example 6 has limit 0 along every straight line approaching (0, 0).

54. If $f(x_0, y_0) = 3$, what can you say about

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y)$$

if f is continuous at (x_0, y_0)? If f is not continuous at (x_0, y_0)? Give reasons for your answers.

The Sandwich Theorem for functions of two variables states that if $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L.$$

Use this result to support your answers to the questions in Exercises 55–58.

55. Does knowing that

$$1 - \frac{x^2 y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

tell you anything about

$$\lim_{(x, y) \to (0,0)} \frac{\tan^{-1} xy}{xy}?$$

Give reasons for your answer.

56. Does knowing that

$$2|xy| - \frac{x^2 y^2}{6} < 4 - 4 \cos \sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x, y) \to (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|}?$$

Give reasons for your answer.

57. Does knowing that $|\sin (1/x)| \leq 1$ tell you anything about

$$\lim_{(x, y) \to (0,0)} y \sin \frac{1}{x}?$$

Give reasons for your answer.

58. Does knowing that $|\cos (1/y)| \leq 1$ tell you anything about

$$\lim_{(x, y) \to (0,0)} x \cos \frac{1}{y}?$$

Give reasons for your answer.

59. (*Continuation of Example 5.*)

a. Reread Example 5. Then substitute $m = \tan \theta$ into the formula

$$f(x, y)\Big|_{y=mx} = \frac{2m}{1 + m^2}$$

and simplify the result to show how the value of f varies with the line's angle of inclination.

b. Use the formula you obtained in part (a) to show that the limit of f as $(x, y) \to (0, 0)$ along the line $y = mx$ varies from -1 to 1 depending on the angle of approach.

60. **Continuous extension** Define $f(0, 0)$ in a way that extends

$$f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

to be continuous at the origin.

Changing to Polar Coordinates If you cannot make any headway with $\lim_{(x, y) \to (0,0)} f(x, y)$ in rectangular coordinates, try changing to polar coordinates. Substitute $x = r \cos \theta$, $y = r \sin \theta$, and investigate the limit of the resulting expression as $r \to 0$. In other words, try to decide whether there exists a number L satisfying the following criterion:

Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad \Rightarrow \quad |f(r, \theta) - L| < \epsilon. \tag{1}$$

If such an L exists, then

$$\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{r \to 0} f(r \cos \theta, r \sin \theta) = L.$$

For instance,

$$\lim_{(x, y) \to (0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r \to 0} \frac{r^3 \cos^3 \theta}{r^2} = \lim_{r \to 0} r \cos^3 \theta = 0.$$

To verify the last of these equalities, we need to show that Equation (1) is satisfied with $f(r, \theta) = r \cos^3 \theta$ and $L = 0$. That is, we need to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad \Rightarrow \quad |r \cos^3 \theta - 0| < \epsilon.$$

Since

$$|r \cos^3 \theta| = |r||\cos^3 \theta| \leq |r| \cdot 1 = |r|,$$

the implication holds for all r and θ if we take $\delta = \epsilon$.

In contrast,

$$\frac{x^2}{x^2 + y^2} = \frac{r^2 \cos^2 \theta}{r^2} = \cos^2 \theta$$

takes on all values from 0 to 1 regardless of how small $|r|$ is, so that $\lim_{(x, y) \to (0,0)} x^2/(x^2 + y^2)$ does not exist.

In each of these instances, the existence or nonexistence of the limit as $r \to 0$ is fairly clear. Shifting to polar coordinates does not always help, however, and may even tempt us to false conclusions. For example, the limit may exist along every straight line (or ray) $\theta =$ constant and yet fail to exist in the broader sense. Example 5 illustrates this point. In polar coordinates, $f(x, y) = (2x^2 y)/(x^4 + y^2)$ becomes

$$f(r \cos \theta, r \sin \theta) = \frac{r \cos \theta \sin 2\theta}{r^2 \cos^4 \theta + \sin^2 \theta}$$

for $r \neq 0$. If we hold θ constant and let $r \to 0$, the limit is 0. On the path $y = x^2$, however, we have $r \sin\theta = r^2\cos^2\theta$ and

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + (r\cos^2\theta)^2}$$

$$= \frac{2r\cos^2\theta\sin\theta}{2r^2\cos^4\theta} = \frac{r\sin\theta}{r^2\cos^2\theta} = 1.$$

In Exercises 61–66, find the limit of f as $(x, y) \to (0, 0)$ or show that the limit does not exist.

61. $f(x, y) = \dfrac{x^3 - xy^2}{x^2 + y^2}$

62. $f(x, y) = \cos\left(\dfrac{x^3 - y^3}{x^2 + y^2}\right)$

63. $f(x, y) = \dfrac{y^2}{x^2 + y^2}$

64. $f(x, y) = \dfrac{2x}{x^2 + x + y^2}$

65. $f(x, y) = \tan^{-1}\left(\dfrac{|x| + |y|}{x^2 + y^2}\right)$

66. $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

In Exercises 67 and 68, define $f(0, 0)$ in a way that extends f to be continuous at the origin.

67. $f(x, y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right)$

68. $f(x, y) = \dfrac{3x^2y}{x^2 + y^2}$

Using the Limit Definition

Each of Exercises 69–74 gives a function $f(x, y)$ and a positive number ϵ. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y),

$$\sqrt{x^2 + y^2} < \delta \quad \Rightarrow \quad |f(x, y) - f(0, 0)| < \epsilon.$$

69. $f(x, y) = x^2 + y^2, \quad \epsilon = 0.01$

70. $f(x, y) = y/(x^2 + 1), \quad \epsilon = 0.05$

71. $f(x, y) = (x + y)/(x^2 + 1), \quad \epsilon = 0.01$

72. $f(x, y) = (x + y)/(2 + \cos x), \quad \epsilon = 0.02$

73. $f(x, y) = \dfrac{xy^2}{x^2 + y^2}$ and $f(0, 0) = 0, \quad \epsilon = 0.04$

74. $f(x, y) = \dfrac{x^3 + y^4}{x^2 + y^2}$ and $f(0, 0) = 0, \quad \epsilon = 0.02$

Each of Exercises 75–78 gives a function $f(x, y, z)$ and a positive number ϵ. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y, z),

$$\sqrt{x^2 + y^2 + z^2} < \delta \quad \Rightarrow \quad |f(x, y, z) - f(0, 0, 0)| < \epsilon.$$

75. $f(x, y, z) = x^2 + y^2 + z^2, \quad \epsilon = 0.015$

76. $f(x, y, z) = xyz, \quad \epsilon = 0.008$

77. $f(x, y, z) = \dfrac{x + y + z}{x^2 + y^2 + z^2 + 1}, \quad \epsilon = 0.015$

78. $f(x, y, z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \epsilon = 0.03$

79. Show that $f(x, y, z) = x + y - z$ is continuous at every point (x_0, y_0, z_0).

80. Show that $f(x, y, z) = x^2 + y^2 + z^2$ is continuous at the origin.