

MATH 2010A/B Advanced Calculus I
 (2014-2015, First Term)
 Homework 2
 Suggested Solution

Exercises 12.3

7. (a) $\mathbf{v} \cdot \mathbf{u} = 5 \times 2 + 1 \times \sqrt{17} = 10 + \sqrt{17}$
 $|\mathbf{v}| = \sqrt{5^2 + 1^2} = \sqrt{26}$
 $|\mathbf{u}| = \sqrt{2^2 + (\sqrt{17})^2} = \sqrt{21}$
- (b) cosine of angle between \mathbf{v} and \mathbf{u} $\cos \theta = \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|} = \frac{10 + \sqrt{17}}{(\sqrt{26})(\sqrt{21})} = \frac{10 + \sqrt{17}}{\sqrt{546}}$
- (c) the scalar component of \mathbf{u} in the direction of $\mathbf{v} = |\mathbf{u}| \cos \theta = \frac{10 + \sqrt{17}}{\sqrt{26}}$
- (d) the vector $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}|^2} \mathbf{v} = \frac{10 + \sqrt{17}}{26} (5\mathbf{i} + \mathbf{j})$

13. $\vec{AB} = (3, 1), \vec{BC} = (-1, -3)$ and $\vec{AC} = (2, -2)$. $\vec{BA} = (-3, -1), \vec{CB} = (1, 3), \vec{CA} = (-2, 2)$.

$$|\vec{AB}| = |\vec{BA}| = \sqrt{10}, |\vec{BC}| = |\vec{CB}| = \sqrt{10}, |\vec{AC}| = |\vec{CA}| = 2\sqrt{2}$$

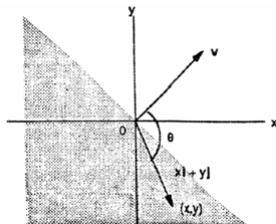
$$\text{Angle at A} = \cos^{-1} \left(\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|} \right) = \cos^{-1} \left(\frac{3(2) + 1(-2)}{(\sqrt{10})(2\sqrt{2})} \right) = \cos^{-1} \left(\frac{1}{\sqrt{5}} \right) \approx 63.435^\circ$$

$$\text{Angle at B} = \cos^{-1} \left(\frac{\vec{BC} \cdot \vec{BA}}{|\vec{BC}||\vec{BA}|} \right) = \cos^{-1} \left(\frac{3}{5} \right) \approx 53.130^\circ$$

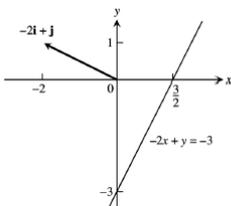
$$\text{Angle at C} = \cos^{-1} \left(\frac{\vec{CB} \cdot \vec{CA}}{|\vec{CB}||\vec{CA}|} \right) = \cos^{-1} \left(\frac{1}{\sqrt{5}} \right) \approx 63.435^\circ$$

18. $\vec{CA} \cdot \vec{CB} = (-\mathbf{v} + (-\mathbf{u})) \cdot (-\mathbf{v} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$
 because $|\mathbf{u}| = |\mathbf{v}|$ since both equal to the radius of the circle. Therefore, \vec{CA} and \vec{CB} are orthogonal.

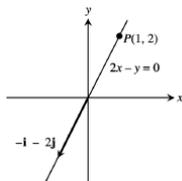
26. $(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{v} = |x\mathbf{i} + y\mathbf{j}||\mathbf{v}| \cos \theta \leq 0$ when $\frac{\pi}{2} \leq \theta \leq \pi$. This means (x, y) has to be a point whose position vector makes an angle with \mathbf{v} greater or equals to 90° .



28. No, $\mathbf{v}_1, \mathbf{v}_2$ need not to be the same. For instance, $2\mathbf{i} + \mathbf{j} \neq \mathbf{i} + \mathbf{j}$ but $(2\mathbf{i} + \mathbf{j}) \cdot \mathbf{j} = 1 = (\mathbf{i} + \mathbf{j}) \cdot \mathbf{j}$.
31. If $a \neq 0$, then the slope of \mathbf{v} is $\frac{b}{a}$ and the slope of $ax + by = c$ is $-\frac{a}{b}$, so the slope of the vector \mathbf{v} is the negative reciprocal of the slope of the given line. If $a = 0$, then $\mathbf{v} = b\mathbf{j}$ is perpendicular to the horizontal line $by = c$. In either case, the vector \mathbf{v} is perpendicular to the line $ax + by = c$.
32. If $a \neq 0$, then the slope of \mathbf{v} is $\frac{b}{a}$ and the slope of $bx - ay = c$ is $\frac{b}{a}$. If $a = 0$, then $\mathbf{v} = b\mathbf{j}$ is parallel to the vertical line $bx = c$. In either case, the vector \mathbf{v} is parallel to the line $bx - ay = c$.
35. $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ is perpendicular to the line $-2x + y = c$; $P(-2, -7)$ is on the line $\Rightarrow -2(-2) - 7 = c \Rightarrow -2x + y = -3$.



39. $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x + y = c$; $P(1, 2)$ is on the line $\Rightarrow -2(1) + 2 = c \Rightarrow -2x - y = 0$ or $2x - y = 0$.



49. $\mathbf{n}_1 = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \mathbf{j}$.
 $\theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{3 + 4}{(\sqrt{25})(\sqrt{2})} \right) = \cos^{-1} \left(\frac{7}{5\sqrt{2}} \right) = 0.14 \text{ rad.}$

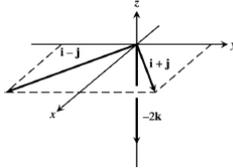
Exercises 12.4

7. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = 6\mathbf{i} - 12\mathbf{k}.$

length $= 6\sqrt{5}$ and the direction is $\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$

$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -6\mathbf{i} + 12\mathbf{k}$, length $= 6\sqrt{5}$ and the direction is $-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$

$$13. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -2\mathbf{k}$$



$$17. (a) \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j}$$

$$\text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{1+1} = \frac{\sqrt{2}}{2}$$

$$(b) \mathbf{u} = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$$

$$21. |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = |-7| = 7$$

23. (a) $\mathbf{u} \cdot \mathbf{v} = -6$, $\mathbf{u} \cdot \mathbf{w} = -81$, $\mathbf{v} \cdot \mathbf{w} = 18$. Therefore, none are perpendicular.

$$(b) \mathbf{u} \times \mathbf{v} = \text{abs} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix} \neq 0$$

$$\mathbf{u} \times \mathbf{w} = \text{abs} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} = 0$$

$$\mathbf{v} \times \mathbf{w} = \text{abs} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{vmatrix} \neq 0$$

Therefore, \mathbf{u} and \mathbf{w} are parallel.

27. (a) always true, $|\mathbf{u}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

(b) not always true, $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

$$(c) \text{ always true, } \mathbf{u} \times \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ 0 & 0 & 0 \end{vmatrix} = 0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ u_1 & u_2 & u_3 \end{vmatrix} = \mathbf{0} \times \mathbf{u}$$

(d) always true,

$$\begin{aligned}\mathbf{u} \times (-\mathbf{u}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ -u_1 & -u_2 & -u_3 \end{vmatrix} \\ &= (-u_2u_3 + u_2u_3)\mathbf{i} - (-u_1u_3 + u_1u_3)\mathbf{j} + (-u_1u_2 + u_1u_2)\mathbf{k} \\ &= \mathbf{0}\end{aligned}$$

(e) not always true, counter example, $\mathbf{j} \times \mathbf{k} = \mathbf{i} \neq -\mathbf{i} = \mathbf{k} \times \mathbf{j}$

(f) always true, distributive property of the cross product

(g) always true, $(\mathbf{u} \times \mathbf{v})$ is always parallel to \mathbf{v} , thus $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

(h) always true, the volume of a parallelepiped with \mathbf{u} , \mathbf{v} and \mathbf{w} along the three edges is the same whether the plane containing \mathbf{u} and \mathbf{v} or the plane containing \mathbf{v} and \mathbf{w} is used as the base plane, and the dot product is commutative.

28. (a) always true, $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v} \cdot \mathbf{u}$

(b) always true, $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = -(\mathbf{v} \times \mathbf{u})$

(c) always true, $(-\mathbf{u}) \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -u_1 & -u_2 & -u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -(\mathbf{u} \times \mathbf{v})$

(d) always true,

$$\begin{aligned}(\mathbf{c}\mathbf{u}) \cdot \mathbf{v} &= (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 \\ &= u_1(cv_1) + u_2(cv_2) + u_3(cv_3) \\ &= \mathbf{u} \cdot (\mathbf{c}\mathbf{v}) \\ &= c(u_1v_1 + u_2v_2 + u_3v_3) \\ &= c(\mathbf{u} \cdot \mathbf{v})\end{aligned}$$

(e) always true,

$$\begin{aligned}c(\mathbf{u}) \times \mathbf{v} &= c \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (\mathbf{c}\mathbf{u}) \times \mathbf{v} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ cv_1 & cv_2 & cv_3 \end{vmatrix} \\ &= \mathbf{u} \times (\mathbf{c}\mathbf{v})\end{aligned}$$

(f) always true, $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2 = (\sqrt{u_1^2 + u_2^2 + u_3^2})^2 = |\mathbf{u}|^2$

(g) always true, $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = \mathbf{0} \cdot \mathbf{u} = 0$

(h) always true, $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v} \Rightarrow (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

39. $\vec{AB} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\vec{DC} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \Rightarrow \vec{AB}$ is parallel to \vec{DC} ; $\vec{BC} = 2\mathbf{i} - \mathbf{j}$ and $\vec{AD} = 2\mathbf{i} - \mathbf{j} \Rightarrow \vec{BC}$ is parallel to \vec{AD} .

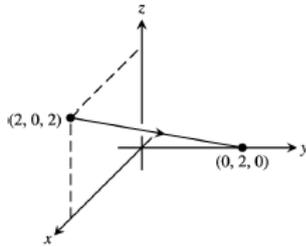
$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 2 & -1 & 0 \end{vmatrix} = 4\mathbf{i} + 8\mathbf{j} - 7\mathbf{k} \Rightarrow \text{Area} = |\vec{AB} \times \vec{BC}| = \sqrt{129}$$

48. $\vec{AB} = \mathbf{i} + 2\mathbf{j}$, $\vec{AC} = -3\mathbf{i} + 2\mathbf{k}$ and $\vec{AD} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

$$\text{Therefore, } (\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix} = 5 \Rightarrow \text{Volume} = |(\vec{AB} \times \vec{AC}) \cdot \vec{AD}| = 5$$

Exercises 12.5

3. The direction $\vec{PQ} = 5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$ and $P(-2, 0, 3) \rightarrow x = -2 + 5t, y = 5t, z = 3 - 5t$
7. The direction \mathbf{k} and $P(1, 1, 1) \Rightarrow x = 1, y = 1, z = 1 + t$
9. The direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $P(0, -7, 0) \Rightarrow x = t, y = -7 + 2t, z = 2t$
19. The direction $\vec{PQ} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $P(2, 0, 2) \Rightarrow x = 2 - 2t, y = 2t, z = 2 - 2t$, where $0 \leq t \leq 1$



23. Let $P = (1, 1, -1)$, $Q = (2, 0, 2)$ and $S = (0, -2, 1)$. Then $\vec{PQ} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\vec{PS} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \Rightarrow \vec{PQ} \times \vec{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ is normal to the plane.

$$\text{Therefore, } 7(x - 2) + (-5)(y - 0) + (-4)(z - 2) = 0 \Rightarrow 7x - 5y - 4z = 6$$

25. $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $P_0(2, 4, 5) \Rightarrow (1)(x - 2) + (3)(y - 4) + (4)(z - 5) = 0 \Rightarrow x + 3y + 4z = 34$

27.

$$\begin{cases} x = 2t + 1 = s + 2 \\ 6 = 3t + 2 = 2s + 4 \end{cases} \Rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases} \Rightarrow t = 0 \text{ and } s = -1$$

Then $z = 4t + 3 = -4s - 1 \Rightarrow 4(0) + 3 = (-4)(-1) - 1$ is satisfied \Rightarrow the lines intersect when $t = 0$ and $s = -1 \Rightarrow$ the point of intersection is $x = 1, y = 2$ and $z = 3$ or $P(1, 2, 3)$.

A vector normal to the plane determined by these lines is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\mathbf{i} + 12\mathbf{j} + \mathbf{k}$, when \mathbf{n}_1 and \mathbf{n}_2 are directions of the lines \Rightarrow the plane containing the lines is represented by $(-20)(x - 1) + (12)(y - 2) + (1)(z - 3) = 0 \Rightarrow -20x + 12y + z = 7$.

29. The cross product of $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $-4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ has the same direction as the normal to the plane

$\Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\mathbf{j} + 6\mathbf{k}$. Select a point on either line, such as $P(-1, 2, 1)$. Since the lines are given to intersect, the desired plane is $(x + 1) + 6(y - 2) + 6(z - 1) = 0 \Rightarrow 6y + 6z = 18 \Rightarrow y + z = 3$.

31. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ is a vector in the direction of the line of intersection of the planes $\Rightarrow 3(x - 2) + (-3)(y - 1) + 3(z + 1) = 0 \Rightarrow 3x - 3y + 3z = 0 \Rightarrow x - y + z = 0$ is the desired plane containing $P_0(2, 1, -1)$.

37. $S(2, 1, -1), P(0, 1, 0)$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$.

Therefore, $d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{4 + 36 + 16}}{\sqrt{4 + 4 + 4}} = \sqrt{\frac{14}{3}}$ is the distance from S to the line.

43. $S(2, 2, 3), 2x + y + 2z = 4$ and $P(2, 0, 0)$ is on the plane $\Rightarrow \vec{PS} = 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

Therefore, $d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{2 + 6}{\sqrt{4 + 1 + 4}} \right| = \frac{8}{3}$

45. The point $P(1, 0, 0)$ is on the first plane and $S(10, 0, 0)$ is a point on the second plane $\Rightarrow \vec{PS} = 9\mathbf{i}$, and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the first plane \Rightarrow the distance from S to the first plane is

$$d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \frac{9}{\sqrt{41}}$$

which is also the distance between the planes.

53. $2x - y + 3z = 6 \Rightarrow 2(1 - t) - (3t) + 3(1 + t) = 6 \Rightarrow -2t + 5 = 6 \Rightarrow t = -\frac{1}{2} \Rightarrow x = \frac{3}{2}, y = -\frac{3}{2}$
and $z = \frac{1}{2} \Rightarrow \left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2} \right)$ is the point.

61. L1 & L2: $x = 3 + 2t = 1 + 4s$ and $y = -1 + 4t = 1 + 2s \Rightarrow s = 1$ and $t = 1$. Therefore, L1 and L2 intersect at $(5, 3, 1)$

L2 & L3: The direction of L2 is $\frac{1}{6}(4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ which is the same as the direction $\frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ of L3. Therefore, L2 and L3 are parallel.

L1 & L3: $x = 3 + 2t = 3 + 2r$ and $y = -1 + 4t = 1 + 2r \Rightarrow t = 1$ and $r = 1 \Rightarrow$ on L1, $z = 2$ but on L3 $z = 0 \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{21}}(2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ while the direction of L3 is $\frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and neither is a multiple of the other; hence L1 and L3 are skew.

67. With substitution of the line into the plane we have $2(1 - 2t) + (2 + 5t) - (-3t) = 8 \Rightarrow 2 - 4t + 2 + 5t + 3t = 8 \Rightarrow 4t + 4 = 8 \Rightarrow t = 1 \Rightarrow$ the point $(-1, 7, -3)$ is contained in both the line and plane, so they are not parallel.

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