**Week 12**: Symmetric Bilinear Forms (textbook § 6.8)

**Recall**: A bilinear form $H: V \times V \to \mathbb{F}$ is symmetric if

$$\boxed{H(x,y) = H(y,x) \quad \text{for all } x, y \in V}$$

Assume $V$ is finite dimensional.

Given any basis $\beta$ for $V$, say $\beta = \{v_1, v_2, \ldots, v_n\}$.

we have a matrix representation of $H$ (w.r.t. $\beta$):

$$\psi_\beta(H) = (H(v_i, v_j)) \in M_{n\times n}(\mathbb{F})$$

**Note**: $H$ symmetric $\Leftrightarrow$ $\psi_\beta(H)$ symmetric (i.e. $A^t = A$)
(for any basis $\beta$)

## "Diagonalization" of Symmetric Bilinear Forms

**Def$^{\underline{n}}$**: $H$ is diagonalizable if there exists some basis $\beta$ for $V$ s.t.

$\psi_\beta(H)$ is diagonal.

In terms of matrices, it is the same as the following:

**Question**: Given $A \in M_{n\times n}(\mathbb{F})$, does there exist an invertible $Q \in M_{n\times n}(\mathbb{F})$ s.t. $Q^t A Q$ is diagonal?

Caution: This is different from the usual notion of diagonalizing a matrix (unless $Q^t = Q^{-1}$) !!

☆ **Theorem**: Let $H$ be a bilinear form over $V$.

$H$ is diagonalizable $\Longleftrightarrow$ $H$ is symmetric ☆

This holds for any field $\mathbb{F}$ (which is not of char. 2).

## How do we understand this theorem?

Two important cases we mostly care about:

$\boxed{\mathbb{F} = \mathbb{R}}$ & $\boxed{\mathbb{F} = \mathbb{C}}$

**Case 1**: $\mathbb{F} = \mathbb{R}$

This is related to the "Real Spectral Theorem", which says (in matrix form) for any $A \in M_{n\times n}(\mathbb{R})$ symmetric, there exists an orthogonal matrix $Q \in M_{n\times n}(\mathbb{R})$ s.t. $Q^{-1}AQ = Q^tAQ$ is diagonal.

★ When $\mathbb{F} = \mathbb{R}$, we can diagonalize a symmetric bilinear form $H$ by finding an orthonormal eigenbasis of the symmetric matrix $\psi_\beta(H)$ under any basis $\beta$. ★

**Example**: Suppose $H(x,y) = x^tAy$ is a symmetric bilinear form on $\mathbb{R}^2$

where $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

By direct calculation, $A$ has two distinct eigenvalues $3$ and $-1$. And $\beta = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is an orthonormal eigenbasis. Thus

$$Q^tAQ = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{where} \quad Q = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\text{orthogonal!}}$$

**Note**: If we take $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ← NOT orthogonal

then

$$Q^tAQ = \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix} \leftarrow \text{still diagonal!}$$

**Observation:** When we "diagonalize" a symmetric bilinear form ($\mathbb{F} = \mathbb{R}$)

$$Q^t A Q = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \qquad d_i \neq \text{eigenvalues of } A \quad (\text{unless } Q^t = Q^{-1})$$

Hence, these $d_i$'s are not "invariants" of the bilinear form $H$.
But the number of positive/negative/zero $d_i$'s are!

Case 2: $\mathbb{F} = \mathbb{C}$

In contrast, "Complex Spectral Theorem" does not help here since

$$\begin{array}{c} A \in M_{n\times n}(\mathbb{C}) \\ \text{symmetric} \\ (\text{i.e. } A^t = A) \end{array} \quad \not\Longrightarrow \quad \begin{array}{c} A \in M_{n\times n}(\mathbb{C}) \\ \text{normal} \\ (\text{i.e. } A^* A = A A^*) \end{array}$$

Even if $A$ is normal, this is still not helpful since we simply get

$$Q^* A Q = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \text{but } Q^* \neq Q^t!$$

We need a different method to "diagonalize" a symmetric bilinear form which hopefully works for any $\mathbb{F}$.

Idea: Apply elementary row/column operations to $A$ simultaneously until it becomes a diagonal matrix!

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow{c2-2\cdot c1} \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \xrightarrow{r2-2\cdot r1} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \quad \text{diagonal}$$

in terms of "elementary matrices":

$$\underbrace{\overset{Q^t}{\overset{\parallel}{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^t}}}_{r2-2\cdot r1} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \underbrace{\overset{Q}{\overset{\parallel}{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}}}}_{c2-2\cdot c1} = \begin{pmatrix} 1 & \\ & -3 \end{pmatrix} \qquad \text{DONE!}$$

## A Complex Example:

$$A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \xrightarrow{c2 - i \cdot c1} \begin{pmatrix} 1 & 0 \\ i & 2 \end{pmatrix} \xrightarrow{r2 - i \cdot r1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Thus,

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}}_{Q^t \quad A \quad Q} = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix}$$

**DONE!**

## A 3×3 Example:

$$A = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix} \xrightarrow{c2 + c1} \begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix} \xrightarrow{r2 + r1} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$$

$$\xrightarrow{c3 - 3 \cdot c1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 3 & 4 & -8 \end{pmatrix} \xrightarrow{r3 - 3 \cdot r1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & -8 \end{pmatrix}$$

$$\xrightarrow{c3 - 4 \cdot c2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & -24 \end{pmatrix} \xrightarrow{r3 - 4 \cdot r2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -24 \end{pmatrix}$$

diagonal!

Therefore, keeping track of all the elementary matrices:

$$Q^t \begin{pmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}}_{= \; Q \; = \begin{pmatrix} 1 & 1 & -7 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -24 \end{pmatrix}$$

**DONE!**

**Note**: One can do the procedure above with less bookkeeping:

$$[A \mid I] \xrightarrow[\text{operations}]{\text{row/column}} [\underset{\substack{\uparrow \\ \text{diagonal}}}{D} \mid Q^t]$$

Ex: Why does it work?

**Example:** $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{c3 + c1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r3 + r1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{c3 - \frac{1}{2}\cdot c2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{r3 - \frac{1}{2}\cdot r2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3/2 & 1 & -1/2 & 1 \end{array}\right] = [D \mid Q^t]$$

Therefore

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}^t}_{= Q^t} \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}}_{= A} \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}}_{= Q} = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3/2 \end{pmatrix}.$$

$\square$

## Invariants of Symmetric Bilinear Forms

Recall that when $T: V \to V$ is a linear operator on a finite dim'l vector space $V$, suppose $\beta$ is an eigenbasis for $T$, then

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \text{when } \lambda_i = \text{eigenvalues of } T$$

indep. of $\beta$ (i.e. invariants of $T$)

In contrast, for a symmetric bilinear form $H$ on $V$, if $\beta$ is a basis for $V$ s.t. $\psi_\beta(H)$ is diagonal, i.e.

$$\psi_\beta(H) = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$$

Then, these diagonal entries $d_i$'s are NOT invariants of $H$, i.e. they depend on the choice of $\beta$.

Let's analyze more carefully......

**Case 1: $\mathbb{F} = \mathbb{R}$**

Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for $V$ s.t.

$$\psi_\beta(H) = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}$$

WLOG, assume $d_1 \geq d_2 \geq \cdots \geq d_n$.

The equality above is equivalent to $\begin{cases} H(v_i, v_i) = d_i \\ H(v_i, v_j) = 0 & \text{for } i \neq j \end{cases}$

If we define a new basis $\beta' = \{v_1', v_2', \dots, v_n'\}$ by

$$\begin{cases} v_i' = \frac{1}{\sqrt{d_i}} v_i & \text{if } d_i > 0 \\ v_i' = v_i & \text{if } d_i = 0 \\ v_i' = \frac{1}{\sqrt{-d_i}} v_i & \text{if } d_i < 0 \end{cases}$$

Then,

$$\psi_{\beta'}(H) = \begin{pmatrix} \boxed{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}} & & \\ & \boxed{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}} & \\ & & \boxed{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}} \end{pmatrix}$$

**Theorem:** Any symmetric bilinear form $H$ on a finite dim'l real vector space can be put into the standard form above. Moreover, the number of $+1, 0, -1$'s are invariants (i.e. indep. of choice of basis) for $H$.

**Example:** If $H$ is an inner product on a real inner product space $V$

then $$\psi_\beta(H) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$ for ANY orthonormal basis $\beta$

**Example:** In Einstein's relativity theory, we consider "inner product" which is not positive definite but has the standard form

$$\begin{array}{r} \text{space} \\ \\ \text{time} \end{array} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

Lorentzian signature

## Case 2: $\mathbb{F} = \mathbb{C}$

Since any non-zero complex number has a square root, the situation is more uniform for $\mathbb{F} = \mathbb{C}$

**Theorem:** Any symmetric bilinear form $H$ on a complex vector space $V$ has a basis for $V$, say $\beta$, s.t

$$\psi_\beta(H) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}.$$

Moreover, the number of 1's and 0's are invariants of $H$ which are independant of the choice of $\beta$.

E.g. Let $H(x,y) = x^T A y$ for $x, y \in \mathbb{C}^2$

where $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If we let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix} \right\}$, then $\psi_\beta(H) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

"Proof of Theorem": The only part we haven't cover is that the number of 0's & 1's are invariants. This follows from the fact that $\text{rank}(A) = \text{rank}(QA) = \text{rank}(AQ)$ for any square matrices $A$ and $Q$ if $Q$ is invertible! Therefore,

$$\text{rank}(Q^t A Q) = \text{rank}(A) = \#\text{ of } 1\text{'s}$$

$$\Rightarrow \text{nullity}(Q^t A Q) = \text{nullity}(A) = \#\text{ of } 0\text{'s}$$

$\square$

The proof for the case $\mathbb{F} = \mathbb{R}$ is similar:

$$\#\text{ of } 1\text{'s} = \max\{\dim W \mid H \text{ is positive definite on } W < V\}$$

ie $H(x,x) > 0$ for all $0 \neq x \in W$.